Computational Topology of Equivariant Maps from Spheres to Complements of Arrangements

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1 Introduction

1.1 Computational Equivariant Topology

Perhaps one of the main general (unsolved) problems of Computational Topology is to determine the scope of the field and major lines of prospective research. This includes the identification of classes of "model problems" where topology and computational mathematics interact in an essential way. The following references may serve as a source of initial information about Computational Topology and as a guide for some of the existing applications.

(ATCS) G. Carlsson (ed.) *Proceedings of the conference on Algebraic Topological Methods in Computer Science*. To appear in the journal Homology, Homotopy and Applications.

(CTOP) M. Bernet. al. Emerging challenges in computational topology. *ACM Computing Research Repository.* arXive: cs.CG/9909001.

(TM) R. Živaljević. Topological Methods. Chapter 14 of *Handbook of Discrete and Computational Geometry* (J.E. Goodman, J. O'Rourke, eds.), new edition, CRC Press, Boca Raton 2004.

One of our objectives in this paper is to identify the problem of calculating the *topological obstructions for the existence of equivariant maps* as one of the problems paradigmatic for computational topology. Recall that, given a group G, a G-equivariant map $f : X \to Y$ between two G-spaces is a symmetry preserving map, i.e. a map satisfying the condition $f(g \cdot x) = g \cdot f(x)$. In applications in discrete and computational geometry, X is usually a manifold of all "feasible" configurations (the configurations space) while Y is typically a complement of a real, affine subspace arrangement (the test space), see (TM).

The problem of calculating/evaluating the complexity of equivariant obstructions has many aspects relevant for computational topology, including the following.

(A_1) The (non)existence of an equivariant map is an essential ingredient in the application of the *configuration space/test map*-scheme (see (TM)), which has proven to be a very effective tool in solving combinatorial or discrete geometric problems of relevance to computing and analysis of algorithms, [14] [28].

(A_2) The existence of an equivariant map can often be interpreted as a problem of mapping an object (often a cell of some dimension) to an Euclidean space, subject to some boundary constraints and avoiding some obstacles (arrangement of subspaces). This aspect can be seen as a relative of the motion planning problem from robotics (the case of a 1-dimensional cell). Recall that the existence of an equivariant map is equivalent to the problem of sectioning a (vector) bundle and the condition of avoiding obstacles is translated into the question of sectioning a bundle subject to some additional constraints (relations). The latter problem is of utmost importance in many areas of mathematics, including Gromov convex integration theory, combinatorial geometry on vector bundles [26] etc.

Conclusion: Both positive and negative aspect of the problem of the existence of an equivariant map is of theoretical interest. In the negative case, i.e. if an equivariant map exists, one should be able to describe and evaluate the complexity of an algorithm that constructs such a map. One way to achieve this goal is to develop an "effective" obstruction theory. This is of particular interest in the context of (A_2) where such a theory would allow effective placements of objects into an environment with obstacles, subject to some boundary constraints.

This paper: We use a well known problem in discrete and computational geometry (Problem 1) as a motivation and an associated question from equivariant topology (Problem 3) as a point of departure to illustrate many aspects, both theoretical and computational, of the general *existence of en equivariant map* problem. A variety of techniques are introduced and discussed with the emphasis on concrete and explicit calculations. This eventually leads (Theorems 18 and 19) to an almost exhaustive analysis of when such maps do or do not exist in this particular case of interest.

1.2 Obstructions to the existence of equivariant maps

Suppose that X and Y are topological spaces and let G be a groups acting on both of them. Suppose that Z is a closed, G-invariant subspace of Y. We focus our attention in this paper on spaces which are simplicial or CW-complexes and finite groups G, see [14] or (TM) for a glossary of basic topological terms. The problem of deciding if there exists a G-equivariant map $f : X \to Y \setminus Z$ can be approached by two, closely related ways. The first is to build such a map step by step, defining it on the skeletons $X^{(k)}$ one at a time, and attempting to extend it to next skeleton $X^{(k+1)}$. The second is to start with a sufficiently generic, equivariant map $f : X \to Y$ and, in the case the singularity $S(f) := f^{-1}(Z)$ is nonempty, try to modify f in attempt to make the set $S(f)$ vanish. In both approaches there may appear *obstructions* which prevent us from completing the process, thus showing that such equivariant maps $f : X \to Y \setminus Z$ do not exist. In the first approach the obstruction is evaluated in the corresponding *equivariant cohomology group*, while in the second it lies in a "dual" *equivariant homology group*, see Section 3.1 for some technical details and references.

2 The motivating problem

Many aspects of *the existence of equivariant map problem* relevant for computational topology are particularly well illustrated by the problem of finding k-fan partitions of spherical measures, see Problem 1 and original references [3], [4] and [23]. Recall that this question arose in connection with some partition problems in discrete and computational geometry, [1], [5], [11], [12], [19] [22]. It turns out that Problem 1 is closely related to a problem of the existence of equivariant maps, see Problem 3 in Section 2.2.

In this paper we give a fairly complete analysis, Theorems 18 and 19, of the Problem 3 and solve it in all but a few exceptional cases.

2.1 Partition of measures by k-fans

A k-fan $\mathfrak{p} = (x; l_1, l_2, \ldots, l_k)$ on the sphere S^2 is a point x, called the center of the fan, and k great semicircles $l_1, \ldots, l_k$ emanating from x. We always assume counter clockwise enumeration of great semicircles $l_1, \ldots, l_k$ of a k-fan. Sometimes we use the notation $\mathfrak{p} = (x; \sigma_1, \sigma_2, \ldots, \sigma_k)$, where σ_i denotes the open angular sector between l_i and l_{i+1}, $i = 1, \ldots, k$.

Let $\mu_1, \mu_2, \ldots, \mu_m$ be *proper* Borel probability measures on S^2. Measure μ is *proper* if $\mu([a, b]) = 0$ for any circular arc $[a, b] \subset S^2$ and $\mu(U) > 0$ for each nonempty open set $U \subset S^2$. All the results can be extended to more general measures, including the counting measures of finite sets, see [3], [25], [26] for related examples.

Let $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{R}^k_{>0}$ be a vector where $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 1$. Following [3], and keeping in mind that we deal only with proper measures, we say that a k-fan $(x; l_1, \ldots, l_k)$ is an *α-partition* for the collection $\{\mu_j\}_{j=1}^m$ of measures if

$$(\forall i = 1, \ldots, k)\, (\forall j = 1, \ldots, m)\, \mu_j(\sigma_i) = \alpha_i.$$

As in [23], a vector $\alpha \in \mathbb{R}^k$ is called *(m, k)-admissible*, if for any collection of m (proper) measures on S^2, there exists a simultaneous α-partition. The collection of all (m, k)-admissible vectors is denoted by $\mathcal{A}_{m,k}$. Here is the central problem about partitions of measures by k-fans.

Problem 1 ([3], [4], [23]) *Describe the set $\mathcal{A}_{m,k}$ or equivalently, find integers m, k and vectors $\alpha \in \mathbb{R}^k$ such that for any collection $\mathcal{M} = \{\mu_1, \mu_2, \ldots, \mu_m\}$ of m (proper) measures , there exist an α-partition for $\mathcal{M}$.*

The analysis given in [3] shows that the most interesting cases are $(3, 2), (2, 3), (2, 4)$. It was shown ([3], [23]) that $\{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3})\} \subseteq \mathcal{A}_{3,2}$ and $\{(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}),\, (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})\} \subseteq \mathcal{A}_{2,4}$.

2.2 The configuration space / test map scheme

The configuration space/test map scheme is a fairly general method of translating combinatorial geometric problems into topological problems, more precisely problems involving equivariant maps, see (TM) or references [14], [25], [28]. It was demonstrated in [3] that the problem of α-partitions of spherical measures also admits such a translation. Recall the key steps of this elegant construction.

Let μ and ν be two Borel, probability measures on S^2 and F_k the space of all k-fans on the sphere S^2. The space X_μ of all n-equipartitions of the measure μ is defined by

$$X_\mu = \{(x; l_1, \ldots, l_n) \in F_n \mid (\forall i = 1, \ldots, n)\, \mu(\sigma_i) = \tfrac{1}{n}\}.$$

Observe that every n-fan $(x; l_1, \ldots, l_n) \in X_\mu$ is completely determined by the pair (x, l_1) or equivalently the pair (x, y), where y is the unit tangent vector to l_1 at x. Thus, the space X_μ is Stiefel manifold $V_2(\mathbb{R}^3)$ of all orthonormal 2-frames in $\mathbb{R}^3$. Recall that $V_2(\mathbb{R}^3) \cong SO(3) \cong \mathbb{R}P^3$.

Let $\mathbb{R}^n$ be an Euclidean space with the standard orthonormal basis $e_1, e_2, \ldots, e_n$ and the associated coordinate functions $x_1, x_2, \ldots, x_n$. Let W_n be the hyperplane $\{x \in \mathbb{R}^n \mid x_1 + x_2 + \ldots + x_n = 0\}$ in $\mathbb{R}^n$ and suppose that the α-vector has the form $\alpha = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}, \frac{a_4}{n}) \in \frac{1}{n}\mathbb{N}^4 \subset \mathbb{Q}^4$ where $a_1 + a_2 + a_3 + a_4 = n$. Then the test map $F_\nu : X_\mu \to W_n \subset \mathbb{R}^n$ for the measure ν is defined by

$$F_\nu(x, y) = F_\nu(x; l_1, \ldots, l_n) = (\nu(\sigma_1) - \tfrac{1}{n}, \nu(\sigma_2) - \tfrac{1}{n}, \ldots, \nu(\sigma_n) - \tfrac{1}{n}).$$

The natural group of symmetries arising in this problem is the dihedral group D_{2n}. It is interesting and sometimes useful to forget a part of the structure and study the associated problem relative to other subgroups $G \subset D_{2n}$. The associated test subspace is in that case the space $D_G(\alpha) = D(\alpha) := \cup\mathcal{A}(\alpha) \subset W_n$ defined as the union of the smallest G-invariant linear subspace arrangement $\mathcal{A}(\alpha)$ in $\mathbb{R}^n$ containing the linear subspace $L(\alpha) \subset W_n$ defined by

$$L(\alpha) := \{x \in \mathbb{R}^n \mid z_1(x) = z_2(x) = z_3(x) = z_4(x) = 0\}, \tag{1}$$

where

$$\begin{aligned} z_1(x) &= x_1 + x_2 + \ldots + x_{a_1}, & z_2(x) &= x_{a_1+1} + \ldots + x_{a_1+a_2}, \\ z_3(x) &= x_{a_1+a_2+1} + \ldots + x_{a_1+a_2+a_3}, & z_4(x) &= x_{a_1+a_2+a_3+1} + \ldots + x_n. \end{aligned}$$

Of course, our central interest is in the case $G = D_{2n}$. Recall [3] that the action of $D_{2n} = \langle \omega, \varepsilon \mid \omega^n = \varepsilon^2 = 1,\, \omega\varepsilon = \varepsilon\omega^{n-1} \rangle$ on the configuration space X_μ and the test space W_n respectively, is given by

$$\begin{aligned} \omega(x; l_1, \ldots, l_n) &= (x; l_2, \ldots, l_n, l_1), & \varepsilon(x; l_1, \ldots, l_n) &= (-x; l_1, l_n, l_{n-1}, \ldots, l_2), \\ \omega(x_1, \ldots, x_n) &= (x_2, \ldots, x_n, x_1), & \varepsilon(x_1, \ldots, x_n) &= (x_n, \ldots, x_2, x_1). \end{aligned}$$

where $(x; l_1, \ldots, l_n) \in X_\mu$ and $(x_1, \ldots, x_n) \in W_n$. D_{2n} acts also on the complement $M(\alpha) = W_n \setminus D(\alpha)$ since $\mathcal{A}(\alpha)$ is the smallest D_{2n}-invariant, linear subspace arrangement which contains linear subspace $L(\alpha)$. Note that D_{2n}-action on the configuration space X_μ is free. Obviously the test map F_ν is a D_{2n}-equivariant map. Note that the configuration space X_μ is D_{2n}-homeomorphic to the Stiefel manifold $V_2(\mathbb{R}^3)$, which as a D_{2n}-space has the action described by

$$\omega(x,y) = (x, R_x(\frac{2\pi}{n})(y)),\ \varepsilon(x,y) = (-x,y)$$

where $R_x(\theta) : \mathbb{R}^3 \to \mathbb{R}^3$ is the rotation around the axes determined by x through the angle θ. The following proposition is a consequence of this analysis.

Proposition 2 *Let $\alpha = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}, \frac{a_4}{n}) \in \frac{1}{n}\mathbb{N}^4 \subset \mathbb{Q}^4$ be a vector such that $a_1 + a_2 + a_3 + a_4 = n$. Let G be a subgroup of the dihedral group D_{2n}. If there does not exist a G-map $F : V_2(\mathbb{R}^3) \to M_G(\alpha)$, where $M_G(\alpha) := W_n \setminus D_G(\alpha)$, then for any two measures μ and ν on S^2, there always exists a 4-fan which simultaneously α-partitions both μ and ν. In other words the non-existence of such a map implies $\alpha \in \mathcal{A}_{2,4}$.*

Recall that in this paper the emphasis is put on computational equivariant topology and the question of the existence of equivariant maps. Hence, in light of Proposition2, it is natural to focus our attention on the following problem as a problem closely related, albeit not equivalent, to the initial α-partition problem.

Problem 3 *For a given subgroup G of the dihedral group D_{2n}, and the associated G-spaces $V_2(\mathbb{R}^3)$ and $M_G(\alpha)$, find an explicit description of the set*

$$\Lambda_G = \{\alpha \in \tfrac{1}{n}\mathbb{N}^4 \mid \text{ There exists a } G\text{-equivariant map } F : V_2(\mathbb{R}^3) \to M(\alpha)\}.$$

The well known "extension of scalars" equivalence from homological algebra, [7] Section III.3, has a very useful analogue in the category of G-spaces and G-equivariant maps, see the Section 5.1. This equivalence permits us in some cases to change the group and replace the original G-space by a new, more tractable topological space, see Proposition 20. Here we replace the dihedral group D_{2n} by the generalized quaternion group Q_{4n} and the original configuration space $V_2(\mathbb{R}^3)$ by the sphere S^3.

Let $S^3 = S(\mathbb{H}) = Sp(1)$ be the group of all unit quaternions and let $\epsilon = \epsilon_{2n} = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n} \in S(\mathbb{H})$ be a root of unity. Group $\langle\epsilon\rangle$ is a subgroup of $S(\mathbb{H})$ of the order $2n$. Then, the *generalized quaternion group,* [8] p. 253*,* is the subgroup

$$Q_{4n} = \{1, \epsilon, \ldots, \epsilon^{2n-1}, j, \epsilon j, \ldots, \epsilon^{2n-1} j\}$$

of S^3 of order $4n$. Let $H = \{1, \epsilon^n\} = \{1, -1\} \subset Q_{4n}$. Then, it is not hard to prove that the quotient group Q_{4n}/H is isomorphic to the dihedral group D_{2n} of the order $2n$.

Proposition 4 *There exists a D_{2n}-map $G : V_2(\mathbb{R}^3) \to M(\alpha)$ if and only if there exists a Q_{4n}-map $F : S^3 \to M(\alpha)$ where the generalized quaternion group Q_{4n} acts on S^3 as a subgroup of $Sp(1) \cong S^3$, and its action on W_n is described by $\epsilon(x_1, \ldots, x_n) = (x_2, \ldots, x_n, x_1)$ and $j(x_1, \ldots, x_n) = (x_n, \ldots, x_2, x_1)$.*

Proof. Note that $\epsilon^n = j^2$ acts trivially on W_n. Moreover, there is an isomorphism $S^3/H \cong \mathbb{R}P^3 \cong SO(3) \cong V_2(\mathbb{R}^3)$. Finally, the $Q_{4n}/H \cong D_{2n}$ action on $S^3/H \cong V_2(\mathbb{R}^3)$ coincides with the D_{2n} action on $V_2(\mathbb{R}^3)$ described in preceding section. Thus this proposition is a direct consequence of the Proposition 20, Section 5.3. ■

Remark 5 *Note that the Q_{4n}-action on S^3 is free. The Q_{4n}-action on W_n is the restriction of the action on $\mathbb{R}^n$ described by*

$$\epsilon \cdot e_i = e_{(i+1) \bmod n} \text{ and } j \cdot e_i = e_{n-i+1}.$$

where $e_1, .., e_n$ be the standard orthonormal basis in $\mathbb{R}^n$.

In light of Proposition 4, Problem 3 is equivalent to the following problem.

Problem 6 *Describe the set*

$$\Gamma_G = \{\alpha \in \tfrac{1}{n}\,\mathbb{N}^4 \mid \text{ There exists a } G\text{-equivariant map } F : S^3 \to M(\alpha)\}.$$

3 Topological preliminaries

3.1 Obstruction Theory

Let G be a subgroup of the generalized quaternion group Q_{4n}. A fundamental problem is to decide if there exists a G-equivariant map $F : S^3 \to M(\alpha)$. The equivariant obstruction theory, as presented in [10] (see also [28], Sections 4.1–4.3), is a versatile tool for studying this question. Recall that $[X, Y]$ is the set of all homotopy classes of maps $f : X \longrightarrow Y$ while in case both X and Y are G-spaces, $[X, Y]_G$ is the corresponding set of all G-homotopy classes of G-equivariant maps.

Since the space $M = M(\alpha)$ is 1-connected and consequently 2-simple in the sense that $\pi_1(M)$ acts trivially on $\pi_2(M)$, then by [10], SectionII.3, and by Hurewicz theorem, $\pi_2(M) \cong [S^2, M] \cong H_2(M; \mathbb{Z})$ as Q_{4n}-modules. Thus, the relevant part of the obstruction exact sequence, [10], [28], has the following form,

$$[S^3, M]_G \quad \xrightarrow{\theta} \quad \mathrm{Im}\left\{[S^3_{(2)}, M]_G \longrightarrow [S^3_{(1)}, M]_G\right\} \quad \xrightarrow{\tau} \quad H^3_G(S^3, H_2(M; \mathbb{Z})).$$

Here $S^3_{(1)}$ and $S^3_{(2)}$ are respectively the 1 and 2-skeleton of the sphere S^3, relative to some Q_{4n}-invariant simplicial or CW-structure. Our initial choice is the simplicial structure arising from the join decomposition $S^3 = P^{(1)}_{2n} * P^{(2)}_{2n}$ where both $P^{(1)}_{2n}$ and $P^{(2)}_{2n}$ are regular $(2n)$-sided polygons, see Figure 6. More precisely, the vertices of these polygons are respectively, $v_i := \epsilon^i a$ and $w_i := \epsilon^i ja$, $i = 0, 1, \ldots, 2n-1$, where a is a fixed complex number, $|a| = 1$. Since $[S^3_{(1)}, M]_G = \{*\}$ is a one-element set and $[S^3_{(2)}, M]_G$ is nonempty, the obstruction exact sequence reduces to,

$$[S^3, M]_G \quad \longrightarrow \quad \{*\} \quad \xrightarrow{\tau} \quad H^3_G(S^3, H_2(M; \mathbb{Z})).$$

The exactness of this sequence means that the set $[S^3, M]_G \neq \emptyset$ if and only if a special element $\tau(*) \in H^3_G(S^3, H_2(M; \mathbb{Z}))$ is equal to zero. Note that $\tau(*)$ depends only on M.

The class $\tau(*)$ can be evaluated by studying the "singular set" of a general position equivariant map, see [28], Sections 4.1-4.3, for a brief overview. A G-simplicial map $h : S^3 \to W_n$ satisfies a "general position condition" if for each simplex σ in S^3 and any linear space U in the arrangement $\mathcal{A}(\alpha)$

$$h(\sigma) \cap U \neq \emptyset \Rightarrow \dim(\sigma) = \dim(h(\sigma)) = 3, \dim(U) = n - 3,\ h(\sigma) \cap U = \{pt\} \subset \mathrm{int}(h(\sigma)).$$

It is not difficult to check that a generic map is in the general position and that every equivariant, simplicial map can be put in general position by a small perturbation. For each G-map $h : S^3 \to W_n$ in the general position, there is an associated obstruction cocycle

$$c(h) \in C^3_G(S^3, A) = \mathrm{Hom}_G(C_3(S^3), A),\ [c(h)] = \tau(*).$$

If σ is an oriented 3-simplex in S^3, then $c(h)(\sigma) \in H_2(M; \mathbb{Z})$ is the image $h_*([\partial(\sigma)])$ of the fundamental class of $\partial(\sigma) \cong S^2$ by the map $h_* : H_2(\partial(\sigma); \mathbb{Z}) \to H_2(M; \mathbb{Z})$. To find explicit form of the obstruction cocycle $c(h) \in C^3_G(S^3, H_2(M; \mathbb{Z})) = \mathrm{Hom}_G(C_3(S^3), H_2(M; \mathbb{Z}))$ we recall that by definition

$$c(h)(\theta) \neq 0 \iff h(\theta) \cap (\cup\mathcal{A}(\alpha)) \neq \emptyset$$

and $c(h)(\theta) = h_*([\partial\theta]) \in H_2(M; \mathbb{Z})$.

Before we go further, let us record for the future reference an important property of the obstruction cocycle. More details on the restriction and the transfer map can be found in [7] Section III.9.

Proposition 7 *Let G be subgroup of Q_{4n}. The cohomology class of the obstruction cocycle $c(h)$ is a torsion element of the group $H^3_G(S^3, H_2(M;\mathbb{Z}))$.*

Proof. Let H be a subgroup of G. There exists a natural "restriction" map $r : H^3_G(S^3, H_2(M;\mathbb{Z})) \to H^3_H(S^3, H_2(M;\mathbb{Z}))$, which on the cochain level is just the "forgetful map" sending a G-cochain $c \in C^3_G(S^3, H_2(M;\mathbb{Z}))$ to the same cochain interpreted as a H-cochain. It follows from the definition of the obstruction cocycle that $r(c(h))$ is the obstruction cocycle for the extension of a general position H-map h. Moreover there exists a natural map $\tau : H^3_H(S^3, H_2(M;\mathbb{Z})) \to H^3_G(S^3, H_2(M;\mathbb{Z}))$ in the opposite direction called the transfer map. It is known, [7] Section III.9. Proposition 9.5.(ii)), that the composition of the restriction with the transfer is just multiplication by the index $[G : H]$:

$$\begin{array}{ccccc} H^3_G(S^3, H_2(M;\mathbb{Z})) & \to & H^3_H(S^3, H_2(M;\mathbb{Z})) & \to & H^3_G(S^3, H_2(M;\mathbb{Z})) \\ [c(h)_G] & \longmapsto & [c(h)_H] & \longmapsto & [G:H]\cdot[c(h)_G]. \end{array}$$

Note that if H is the trivial group, the cohomology class of the obstruction cocycle $[c(h)_H]$ is zero. This implies that $[G : H] \cdot [c(h)_G] = 0$ in $H^3_G(S^3, H_2(M;\mathbb{Z}))$, i.e. $[c(h)_G]$ is a torsion element of the group $H^3_G(S^3, H_2(M;\mathbb{Z}))$. ■

Remark 8 *The previous result explains why we pay a special attention in subsequent sections to the torsion part of the group $H^3_G(S^3, H_2(M;\mathbb{Z}))$.*

3.2 The Z-Ž formula and point classes of $M(\mathcal{A}(\alpha))$

Before we continue with the calculation of the obstruction cocycle $c_G(h) \in C^3_G(S^3, H_2(M;\mathbb{Z}))$, we should gather together more information about the G-module $H_2(M;\mathbb{Z})$. Since every subspace $K \in \mathcal{A}(\alpha)$ has the codimension in W_n greater or equal to 3, the fundamental group $\pi_1(M)$ of the complement $M = W_n \setminus \cup\mathcal{A}(\alpha)$ is trivial. Thus, $H_1(M;\mathbb{Z}) = 0$ and the first nontrivial homology group of the complement is $H_2(M;\mathbb{Z})$.

Before we begin the analysis of the homology group $H_2(M;\mathbb{Z})$of the complement $M = W_n \setminus \cup\mathcal{A}(\alpha)$, we make a few general observations about representations of homology classes of complements of arrangements. Suppose that $L_1, L_2, \ldots, L_k$ is a collection of a-dimensional linear subspaces in an $(a+b)$-dimensional, Euclidean space V. Let $\mathcal{A}$ be the associated arrangement with the intersection poset $P_{\mathcal{A}}$, compactified union (link) $\hat{D}(\mathcal{A}) = \cup\mathcal{A} \cup \{+\infty\} \subset V \cup \{+\infty\} \cong S^{a+b}$, and the complement $M(\mathcal{A}) = V \setminus \cup\mathcal{A}$. For a given point $x \in L_i \setminus \cup_{j\neq i} L_j$, let $D_\epsilon(x) = x + D_\epsilon$ be a small disc around x, where $D_\epsilon = \{y \in L_i^\perp \mid \langle y, y\rangle \leq \epsilon\}$. "Small" means that ϵ is chosen so that $D_\epsilon(x) \cap (\cup_{j\neq i} L_j) = \emptyset$. We assume that $D_\epsilon(x)$ is oriented, typically by the orientation inherited from some orientations on V and L_i which are prescribed in advance. The fundamental class of the pair $(D_\epsilon(x), \partial D_\epsilon(x))$ determines a homology class in $H_b(V, M(\mathcal{A});\mathbb{Z})$, which we denote by $[x]$ and call the *point class* of x. Note that by the Excision axiom, $[x]$ does not depend on ϵ. Moreover, by the Homotopy axiom, $[x]$ does not change if x is moved inside a connected component of $L_i \setminus \cup_{j\neq i} L_j$. Similarly, in light of the isomorphism $H_b(V, M(\mathcal{A})) \to H_{b-1}(M(\mathcal{A}))$, the class $[[x]] := \partial[x]$, which is also called the *point class* of x, has all these invariance properties as well.

Let us show that the class $[x]$ is always nontrivial. By the Ziegler-Živaljević formula [30], the homotopy type of the one-point compactification $\hat{D}(\mathcal{A}) = \cup\mathcal{A} \cup \{+\infty\}$ has the wedge decomposition of the form,

$$\hat{D}(\mathcal{A}) \simeq \hat{L}_1 \vee \hat{L}_2 \vee \ldots \vee \hat{L}_k \vee \ldots$$

where the displayed factors correspond to elements $p \in P(\mathcal{A})$ of the minimum dimension $d(p) = a$. Let $c(\hat{L}_i) \in H^b(V, M(\mathcal{A});\mathbb{Z})$ be the cohomology class which is Poincaré-Alexander dual of the fundamental homology class $[\hat{L}_i] \in H_a(\hat{D}(\mathcal{A});\mathbb{Z})$ associated with the sphere $\hat{L}_i$ in $\hat{V}$. Then $c(\hat{L}_i)([N]) \in \mathbb{Z}$ is essentially the intersection number $[\hat{l}_i] \cap [N]$, whenever this number is correctly defined, for example if N is a manifold and the intersection $\hat{L}_i \cap N$ is transversal. From here we see that $c(\hat{L}_i)([x]) = 1$, hence $[x]$ must be nontrivial. All this also applies to the class $[[x]] = \partial[x] \in H_{b-1}(M(\mathcal{A}))$. Consequently we have the following proposition.

Proposition 9 *If the codimension of $L_{ij} := L_i \cap L_j$ in L_i is greater than 1 for each $j \neq i$ and $x \in L_i \setminus \cup_{j\neq i} L_j$, then the point class $[x]$ is a well defined class in $H_b(V, M(\mathcal{A}))$ which does not depend on $x \in L_i$ whatsoever.*

What happens when $\mathrm{codim}_{L_i}(L_i \cap L_j) = 1$ for some $j \neq i$? Then L_{ij}, being a hyperplane in both L_i and L_j, decomposes these spaces into the union of closed halfspaces, $L_i = L_i^1 \cup L_i^2$ and $L_j = L_j^1 \cup L_j^2$ respectively. By gluing the halfspaces L_i^1 and L_j^1 along the common boundary L_{ij} and by adding the infinite point $+\infty$, one obtains a sphere $K \subset \hat{V}$. Recall that the decomposition in Ziegler-Živaljević formula [30] involves a choice of generic points in all elements of the arrangement $\mathcal{A}$. The points can be chosen in halfspaces L_i^1 and L_j^1, which implies that the sphere K appears in the decomposition as the factor $\hat{L}_{ij} * \Delta(P_{<p_{ij}}) \cong S^{a-1} * S^0 \cong S^a$, where $p_{ij} \in P$ corresponds to $L_{ij} \in \mathcal{A}$. This guarantees that the fundamental class $[K]$ of K is nontrivial in $H_a(\hat{D}(\mathcal{A}))$. Let $c(K) \in H^b(V, M(\mathcal{A}); \mathbb{Z})$ be the Poincaré-Alexander dual to $[K]$. Then $c(K)([x_1]) = \pm 1$ if x_1 is in the interior of the halfspace L_i^1 but $c(K)([x_2]) = 0$ if x_2 belongs to the interior of the complementary halfspace L_i^2. This observation leads to the following proposition.

Proposition 10 *Let $\mathrm{codim}_{L_i}(L_i \cap L_j) = 1$ for some $j \neq i$ where $L_i = L_i^1 \cup L_i^2$, $L_j = L_j^1 \cup L_j^2$ are decompositions in closed halfspaces. If points x_1 and x_2 belong to interiors of complementary halfspaces L_i^1 and L_i^2then $[x_1] \neq [x_2]$.*

4 Calculation of the obstruction cocycles

Now we are ready to compute the homology class of the cocycle $c_G(h) \in C^3_G(S^3, H_2(M; \mathbb{Z}))$. As before G is a subgroup of the generalized quaternion group. Our primary interest is in the complete group Q_{4n} and the cyclic subgroup $\mathbb{Z}_n = \{1, \epsilon^2, \ldots, \epsilon^{2n-2}\}$. In the latter case we complete the calculations started in [23].

4.1 The obstruction cocycle

4.1.1 General position Q_{4n}-maps and their singular sets

Let us start with the description of a general position, simplicial Q_{4n}-map $h : S^3 \to W_n$, where the sphere has the simplicial structure $S^3 = P_{2n}^{(1)} * P_{2n}^{(2)}$, described in Section 3.1 and depicted in Figure 6. Let $e_1, .., e_n$ be the standard orthonormal basis in $\mathbb{R}^n$ and let $x_1, .., x_n$ be associated dual linear functions. Let $\{u_1, .., u_n\}$, where $u_i = e_i - e$ and $e = \frac{1}{n}\sum_{r=1}^n e_r$, be the vertex set of a regular simplex Δ_{n-1} in W_n. If the map h is prescribed in advance on the vertex $a \in P_{2n}^{(1)} * P_{2n}^{(2)}$, say if $h(a) = u_1$, and if we require that it is a simplicial Q_{4n}-map, then everything else is completely determined. For example,

$$\begin{aligned}
h(\epsilon^i a) &= \epsilon^i \cdot h(a) = \epsilon^i \cdot u_1 = \epsilon^i \cdot (e_1 - e) = e_{i \bmod n+1} - e = u_{i \bmod n+1} \\
h(ja) &= j \cdot h(a) = j \cdot u_1 = j \cdot (e_1 - e) = e_n - e = u_n \\
h(\epsilon^i ja) &= \epsilon^i j \cdot h(a) = \epsilon^i \cdot (e_n - e) = e_{(i+n) \bmod n} - e = e_i - e = u_{i \bmod n}.
\end{aligned}$$

To see that h is a general position map, it is sufficient to test for which simplexes $\sigma = \sigma_1 * \sigma_2 \subset P_{2n} * P_{2n}$ the image $h(\sigma)$ intersects the subspace $L = L(\alpha)$. First we observe that $h(P_{2n} * P_{2n}) \subseteq \mathrm{sk}_3(\Delta_{n-1})$ and that $L \cap h(P_{2n} * P_{2n}) = \{y_1, y_2\}$, where

$$\begin{aligned}
y_1 &= \frac{a_1}{n} u_{a_1} + \frac{a_2}{n} u_{a_1+1} + \frac{a_3}{n} u_{a_1+a_2+a_3} + \frac{a_4}{n} u_{a_1+a_2+a_3+1} \\
y_2 &= \frac{a_2}{n} u_{a_1+a_2} + \frac{a_3}{n} u_{a_1+a_2+1} + \frac{a_4}{n} u_n + \frac{a_1}{n} u_1.
\end{aligned}$$

Thus, there are only two 3-simplices

$$\tau_1 = [u_{a_1}, u_{a_1+1}; u_{a_1+a_2+a_3}, u_{a_1+a_2+a_3+1}] \text{ and } \tau_2 = [u_{a_1+a_2}, u_{a_1+a_2+1}; u_n, u_1]$$

in the simplex Δ_{n-1} which intersect L. To find the singular set $h^{-1}(\cup\mathcal{A}(\alpha)) \subseteq S^3$, we have to detect all 3-simplices $\sigma = \sigma_1 * \sigma_2 \subset P_{2n} * P_{2n}$ in the sphere S^3 with the property $\sigma \cap h^{-1}(L) \neq \emptyset$, or in other words simplices σ such that either $h(\sigma) = \tau_1$ or $h(\sigma) = \tau_2$. This leads to the following systems of equations:

$$
\begin{array}{lll}
h(\epsilon^i a) = u_{i \bmod n+1} = u_{a_1} & \text{and} & h(\epsilon^i ja) = u_{i \bmod n} = u_{a_1+a_2+a_3} \\
h(\epsilon^i a) = u_{i \bmod n+1} = u_{a_1+a_2+a_3} & \text{and} & h(\epsilon^i ja) = u_{i \bmod n} = u_{a_1} \\
h(\epsilon^i a) = u_{i \bmod n+1} = u_{a_1+a_2} & \text{and} & h(\epsilon^i ja) = u_{i \bmod n} = u_n \\
h(\epsilon^i a) = u_{i \bmod n+1} = u_n & \text{and} & h(\epsilon^i ja) = u_{i \bmod n} = u_{a_1+a_2}.
\end{array}
$$

There are 16, not necessarily different, 3-simplices $\theta_1, \ldots, \theta_{16}$ in $P_{2n} * P_{2n}$ which nontrivially intersect the singular subset $h^{-1}(L)$. Here is a complete list where $v_i = \epsilon^i a$, $w_i = \epsilon^i ja$ for $i \in \{0, .., 2n-1\}$ and $P = a_1$, $Q = a_1 + a_2$, $R = a_1 + a_2 + a_3$.

$$
\begin{array}{ll}
\theta_1 = [v_{P-1}, v_P; w_R, w_{R+1}], & \theta_2 = [v_{n+P-1}, v_{n+P}; w_R, w_{R+1}], \\
\theta_3 = [v_{P-1}, v_{a_1}; w_{n+R}, w_{n+R+1}], & \theta_4 = [v_{n+P-1}, v_{n+P}; w_{n+R}, w_{n+R+1}], \\
\theta_5 = [v_{R-1}, v_R; w_P, w_{P+1}], & \theta_6 = [v_{n+R-1}, v_{n+R}; w_P, w_{P+1}], \\
\theta_7 = [v_{R-1}, v_R; w_{n+P}, w_{n+P+1}], & \theta_8 = [v_{n+R-1}, v_{n+R}; w_{n+P}, w_{n+P+1}], \\
\theta_9 = [v_{Q-1}, v_Q; w_n, w_{n+1}], & \theta_{10} = [v_{n+Q-1}, v_{n+Q}; w_n, w_{n+1}], \\
\theta_{11} = [v_{Q-1}, v_Q; w_0, w_1], & \theta_{12} = [v_{n+Q-1}, v_{n+Q}; w_0, w_1], \\
\theta_{13} = [v_{n-1}, v_n; w_Q, w_{Q+1}], & \theta_{14} = [v_{2n-1}, v_0; w_Q, w_{Q+1}], \\
\theta_{15} = [v_{n-1}, v_n; w_{n+Q}, w_{n+Q+1}], & \theta_{16} = [v_{2n-1}, v_0; w_{n+Q}, w_{n+Q+1}].
\end{array}
$$

Note that

$$
h(\theta_i) = \tau_1 \text{ if } i \in \{1, .., 8\} \text{ and } h(\theta_i) = \tau_2 \text{ if } i \in \{9, .., 16\}.
$$

4.1.2 The obstruction cocycle $c_{Q_{4n}}(h)$

We have made all necessary preparations and now we are finally ready to compute the obstruction cocycle $c_{Q_{4n}}(h)$. For a 3-simplex θ in S^3,

$$
c_{Q_{4n}}(h)(\theta) = \sum_{y \in h(\theta) \cap \cup\mathcal{A}(\alpha)} \mathrm{I}(h(\theta), L_y)[y]
$$

where $\mathrm{I}(h(\theta), L_y)$ is the intersection number of the oriented simplex $h(\theta)$ and the appropriate oriented element L_y ($L_y \cap h(\theta) = \{y\}$) of the arrangement $\mathcal{A}(\alpha)$ and $[y] \in H_2(M; \mathbb{Z})$ is a point class.

In order to simplify the computation it is convenient to replace the simplicial complex structure $P_{2n} * P_{2n}$ on S^3 and its chain complex $\{C_i(S^3, \mathbb{Z})\}_{i=0}^3$ by a much more economical Q_{4n}-invariant, cell complex structure on S^3 with the chain complex $\{D_i(S^3, \mathbb{Z})\}_{i=0}^3$, described in the Appendix, Section 5.1. The obstruction class computed relative to the new cell complex structure is denoted by $c'_{Q_{4n}}(h)$.

To evaluate the obstruction cocycle $c'_{Q_{4n}}(h)$ on the cell e, keeping in mind natural maps of chain complexes $\{C_i(S^3, \mathbb{Z})\}_{i=0}^3$ and $\{D_i(S^3, \mathbb{Z})\}_{i=0}^3$, we have to find how many simplexes of the form $g \cdot \theta_i$, where $g \in Q_{4n}$ and $i \in \{1, .., 16\}$, belong to the cell e. It is found by inspection that

$$
\begin{array}{rcl}
\sigma_1 & = & [v_{a_1+a_4-1}, v_{a_1+a_4}; w_0, w_1] = \epsilon^{-a_1-a_2-a_3}\theta_2 = j\epsilon^{-a_1}\theta_1, \\
\sigma_2 & = & [v_{a_2+a_3-1}, v_{a_2+a_3}; w_0, w_1] = \epsilon^{-a_1}\theta_5 = j\epsilon^{-a_1-a_2-a_3}\theta_7, \\
\sigma_3 & = & [v_{a_2+a_1-1}, v_{a_2+a_1}; w_0, w_1] = \theta_{11} = j\epsilon^{-a_1-a_2}\theta_9, \\
\sigma_4 & = & [v_{a_3+a_4-1}, v_{a_3+a_4}; w_0, w_1] = \epsilon^{-a_1-a_2}\theta_{13} = j\theta_{14},
\end{array}
$$

is the complete list of these simplices. In other words there are four (not necessarily different **!**) simplices σ_1, σ_2, σ_3, σ_4 in the cell e such that $h(\sigma_i) \cap D(\alpha) \neq \varnothing$ for each $i \in \{1, 2, 3, 4\}$. Also observe that for each

of the simplices σ_1, σ_2, σ_3, σ_4 the intersection $h(\sigma_i)\cap\, D(\alpha)$ has two (not necessarily different **!**) points. More explicitly these points are

$$\begin{array}{rcl}
x_{11} & = & \frac{a_1}{n}v_{a_1+a_4-1}+\frac{a_2}{n}v_{a_1+a_4}+\frac{a_3}{n}w_0+\frac{a_4}{n}w_1 \in \sigma_1\cap h^{-1}(\epsilon^{-a_1-a_2-a_3}L),\\
x_{12} & = & \frac{a_4}{n}v_{a_1+a_4-1}+\frac{a_3}{n}v_{a_1+a_4}a+\frac{a_2}{n}w_0+\frac{a_1}{n}w_1 \in \sigma_1\cap h^{-1}(j\epsilon^{-a_1}L),\\
x_{21} & = & \frac{a_3}{n}v_{a_2+a_3-1}+\frac{a_4}{n}v_{a_2+a_3}+\frac{a_1}{n}w_0+\frac{a_2}{n}w_1 \in \sigma_2\cap h^{-1}(\epsilon^{-a_1}L),\\
x_{22} & = & \frac{a_2}{n}v_{a_2+a_3-1}+\frac{a_1}{n}v_{a_2+a_3}+\frac{a_4}{n}w_0+\frac{a_3}{n}w_1 \in \sigma_2\cap h^{-1}(j\epsilon^{-a_1-a_2-a_3}L),\\
x_{31} & = & \frac{a_2}{n}v_{a_2+a_1-1}+\frac{a_3}{n}v_{a_2+a_1}+\frac{a_4}{n}w_0+\frac{a_1}{n}w_1 \in \sigma_3\cap h^{-1}(L),\\
x_{32} & = & \frac{a_1}{n}v_{a_2+a_1-1}+\frac{a_4}{n}v_{a_2+a_1}+\frac{a_3}{n}w_0+\frac{a_2}{n}w_1 \in \sigma_3\cap h^{-1}(j\epsilon^{-a_1-a_2}L),\\
x_{41} & = & \frac{a_4}{n}v_{a_3+a_4-1}+\frac{a_1}{n}v_{a_3+a_4}+\frac{a_2}{n}w_0+\frac{a_3}{n}w_1 \in \sigma_4\cap h^{-1}(\epsilon^{-a_1-a_2}L),\\
x_{42} & = & \frac{a_3}{n}v_{a_3+a_4-1}+\frac{a_2}{n}v_{a_3+a_4}+\frac{a_1}{n}w_0+\frac{a_4}{n}w_1 \in \sigma_4\cap h^{-1}(jL),
\end{array}$$

then

$$\begin{array}{rcl}
h(\sigma_1)\cap D(\alpha) & = & (h(\sigma_1)\cap\epsilon^{-a_1-a_2-a_3}L)\cup(h(\sigma_1)\cap j\epsilon^{-a_1}L)=\{h(x_{11}),h(x_{12})\};\\
h(\sigma_2)\cap D(\alpha) & = & (h(\sigma_2)\cap\epsilon^{-a_1}L)\cup(h(\sigma_2)\cap j\epsilon^{-a_1-a_2-a_3}L)=\{h(x_{21}),h(x_{22})\};\\
h(\sigma_3)\cap D(\alpha) & = & (h(\sigma_3)\cap L)\cup(h(\sigma_3)\cap j\epsilon^{-a_1-a_2}L)=\{h(x_{31}),h(x_{32})\};\\
h(\sigma_4)\cap D(\alpha) & = & (h(\sigma_4)\cap\epsilon^{-a_1-a_2}L)\cup(h(\sigma_4)\cap jL)=\{h(x_{41}),h(x_{42})\}
\end{array}$$

and

$$h(e)\cap D(\alpha)=\{h(x_{11}),h(x_{12}),h(x_{21}),h(x_{22}),h(x_{31}),h(x_{32}),h(x_{41}),h(x_{42})\}.$$

Now we focus on the cardinality of the intersection $h(e)\cap D(\alpha)$.As it turns out, the spaces $D(\alpha)$ and the corresponding cardinalities are very sensitive to the change of integers a_1,a_2,a_3,a_4. We pay a special attention to the question whether the test map h is in general position since this is a key technical assumption used in computations. We discuss several separate cases.

(A) Let $a_2=a_4$ and $a_1\neq a_3$, in which case $\sigma_1=\sigma_3$, $\sigma_2=\sigma_4$ and $x_{11}=x_{32}$, $x_{12}=x_{31}$, $x_{21}=x_{42}$, $x_{22}=x_{41}$. These identities imply that

$$\begin{array}{ll}
h(x_{11})=h(x_{32})\in\epsilon^{-a_1-a_2-a_3}L\cap j\epsilon^{-a_1-a_2}L, & h(x_{12})=h(x_{31})\in j\epsilon^{-a_1}L\cap L,\\
h(x_{21})=h(x_{42})\in\epsilon^{-a_1}L\cap jL & h(x_{22})=h(x_{41})\in j\epsilon^{-a_1-a_2-a_3}L\cap\epsilon^{-a_1-a_2}L
\end{array}$$

A question arises whether h is in general position. However, cf. the proof of Theorem 17 (J), there is a relation $\epsilon^{-a_1}L=jL$ which implies

$$\begin{array}{rcl}
j\epsilon^{-a_1-a_2}L & = & \epsilon^{(n-1)(-a_1-a_2)}jL=\epsilon^{a_1+a_2}jL=\epsilon^{a_2}L=\epsilon^{a_4}L=\epsilon^{-a_1-a_2-a_3}L,\\
\epsilon^{-a_1}L & = & jL\\
j\epsilon^{-a_1-a_2-a_3}L & = & j\epsilon^{a_4}L=\epsilon^{(n-1)a_4}jL=\epsilon^{-a_4-a_1}L=\epsilon^{-a_2-a_1}L\\
j\epsilon^{-a_1}L & = & \epsilon^{(n-1)(-a_1)}jL=\epsilon^{a_1}jL=L
\end{array}$$

and shows that h is indeed in general position.

(B) Let $a_2=a_4\neq a_1=a_3$, in which case $\sigma_1=\sigma_3=\sigma_2=\sigma_4$ and $x_{11}=x_{32}=x_{21}=x_{42}$, $x_{12}=x_{31}=x_{22}=x_{41}$. In this case

$$\begin{array}{rcl}
h(x_{11}) & = & h(x_{32})=h(x_{21})=h(x_{42})\in\epsilon^{-a_1-a_2-a_3}L\cap j\epsilon^{-a_1-a_2}L\cap\epsilon^{-a_1}L\cap jL,\\
h(x_{12}) & = & h(x_{31})=h(x_{22})=h(x_{41})\in j\epsilon^{-a_1}L\cap L\cap j\epsilon^{-a_1-a_2-a_3}L\cap\epsilon^{-a_1-a_2}L,
\end{array}$$

and again, there is a question of the genericity of h. The proof of Theorem 17, (B) shows that there exist relations $\epsilon^{a_1+a_2}L=L$ and $\epsilon^{a_2}L=jL$ which again implies that h is in general position

$$\epsilon^{-a_1}L=\epsilon^{a_2}L=jL=j\epsilon^{a_1+a_2}L=j\epsilon^{-a_1-a_2}L=\epsilon^{-a_1-a_2-a_3}L.$$

(C) Let $a_1 = a_3$ and $a_2 \neq a_4$, then $\sigma_1 = \sigma_4$, $\sigma_2 = \sigma_3$ and $x_{11} = x_{42}$, $x_{12} = x_{41}$, $x_{21} = x_{32}$, $x_{22} = x_{31}$. Like in the case (A), it is not difficult to prove that h is again in general position.

(D) Let $a_1 = a_2 = a_3 = a_4$, then $\sigma_1 = \sigma_3 = \sigma_2 = \sigma_4$ and $x_{11} = x_{32} = x_{21} = x_{42} = x_{12} = x_{31} = x_{22} = x_{41}$. Thus,

$$h(x_{11}) = .. = h(x_{41}) \in \epsilon^{-a_1-a_2-a_3}L \cap j\epsilon^{-a_1-a_2}L \cap \epsilon^{-a_1}L \cap jL \cap j\epsilon^{-a_1}L \cap L \cap j\epsilon^{-a_1-a_2-a_3}L \cap \epsilon^{-a_1-a_2}L$$

and again Theorem 17, (A) helps with relations $\epsilon^{a_1}L = L$ and $jL = L$. Therefore

$$j\epsilon^{-a_1-a_2-a_3}L = j\epsilon^{-a_1-a_2}L = j\epsilon^{-a_1}L = jL = L = \epsilon^{-a_1}L = \epsilon^{-a_1-a_2-a_3}L = \epsilon^{-a_1-a_2}L$$

and h is in general position.

(E) Let $a_1 = a_2 \neq a_3 = a_4$, then $\sigma_1 = \sigma_2$ and $x_{11} = x_{22}$, $x_{12} = x_{21}$. In this case

$$\begin{aligned} h(x_{11}) &= h(x_{22}) \in \epsilon^{-a_1-a_2-a_3}L \cap j\epsilon^{-a_1-a_2-a_3}L \\ h(x_{12}) &= h(x_{21}) \in j\epsilon^{-a_1}L \cap \epsilon^{-a_1}L. \end{aligned}$$

With the relations $jL = \epsilon^{2a_3}L$ (Theorem 17, (I)) we have

$$\begin{aligned} j\epsilon^{-a_1-a_2-a_3}L &= j\epsilon^{a_4}L = \epsilon^{(n-1)a_4}jL = \epsilon^{-a_4+2a_3}L = \epsilon^{a_4}L = \epsilon^{-a_1-a_2-a_3}L, \\ j\epsilon^{-a_1} &= \epsilon^{(n-1)(-a_1)}jL = \epsilon^{a_1+2a_3}L = \epsilon^{-a_1}L, \end{aligned}$$

and conclude that h is in general position again.

(F) Let $a_2 = a_3 \neq a_1 = a_4$, then $\sigma_3 = \sigma_4$ and $x_{31} = x_{42}$, $x_{32} = x_{41}$. Like in the preceding case (E), it is not hard to prove that h is in general position.

(G) In all the remaining cases

$$h(e) \cap D(\alpha) = \{h(x_{11}), h(x_{12}), h(x_{21}), h(x_{22}), h(x_{31}), h(x_{32}), h(x_{41}), h(x_{42})\}.$$

Theorem 11 *Let $c'_{Q_{4n}}(h) \in C^3_{Q_{4n}}(S^3, H_2(M;\mathbb{Z}))$ be the (second) obstruction cocycle for the map h defined above. Let K be the additive subgroup of $H_2(M;\mathbb{Z})$ generated by the elements of the form $g \cdot x - x$, $g \in Q_{4n}$, $x \in H_2(M;\mathbb{Z})$. As usual, the group of coinvariants [7] is $H_2(M;\mathbb{Z})_{Q_{4n}} = H_2(M;\mathbb{Z})/K$.*

(A) Let $(a_1, a_2, a_3, a_4) = (p, p, p, p)$. Then $c'_{Q_{4n}}(h)(e) = [[y_2]] + K \in H_2(M;\mathbb{Z})_{Q_{4n}}$.

(B) Let $(a_1, a_2, a_3, a_4) = (p, p, p, 2p)$. Then $[c'_{Q_{4n}}(h)(e)] = 2((1+(-1)^{\binom{5p}{2}}))[[y_1]]+K$ in $H_2(M;\mathbb{Z})_{Q_{4n}}$.

(C) Let $(a_1, a_2, a_3, a_4) \neq (p, p, p, p)$. Then there exists $z \in H_2(M;\mathbb{Z})$ such that $[c'_{Q_{4n}}(h)(e)] = 2z + K$ in $H_2(M;\mathbb{Z})_{Q_{4n}}$.

Proof. Before we calculate obstruction cocycle in all these cases recall that

$$c_{Q_{4n}}(h)(\theta) = \sum_{y \in h(\theta) \cap D(\alpha)} \mathrm{I}(h(\theta), L_y)[y]$$

where $\mathrm{I}(h(\theta), L_y)$ is the intersection number of the oriented simplex $h(\theta)$ and the appropriate oriented element L_y ($L_y \cap h(\theta) = \{y\}$) of the arrangement $\mathcal{A}(\alpha)$. Also, for every $g \in Q_{4n}$ there is an identity

$$\mathrm{I}(h(g\theta), L_{g\cdot y}) = \det(g)\, \mathrm{I}(h(\theta), L_y)$$

where $\det(\epsilon) = (-1)^{n+1}$, $\det(j) = (-1)^{\binom{\binom{n}{2}}{}}$ and $\det(g \cdot g') = \det(g) \cdot \det(g')$. Following calculations from [23], we have:

$$\mathrm{I}(h(\theta_i), L) = \begin{cases} 1 & , i \in \{1, .., 8\} \\ -1 & , i \in \{9, .., 16\} \end{cases}, \text{ where } h(\theta_i) \cap L = h(x_{\theta_i}) = \begin{cases} y_1 & , i \in \{1, .., 8\} \\ y_2 & , i \in \{9, .., 16\} \end{cases}.$$

From the first equation we get

$$c'_{Q_{4n}}(h)(e) = \sum_{\theta\subset e}\ \sum_{y\in h(\theta)\cap D(\alpha)} \mathrm{I}(h(\theta), L_y)[y].$$

Also, observe that

$$\begin{array}{ll}
[[h(x_{11})]] = \epsilon^{-a_1-a_2-a_3}[[y_1]], & [[h(x_{12})]] = j\epsilon^{-a_1}[[y_1]],\\
[[h(x_{21})]] = \epsilon^{-a_1}[[y_1]], & [[h(x_{22})]] = j\epsilon^{-a_1-a_2-a_3}[[y_1]],\\
[[h(x_{31})]] = [[y_2]], & [[h(x_{32})]] = j\epsilon^{-a_1-a_2}[[y_2]],\\
[[h(x_{41})]] = \epsilon^{-a_1-a_2}[[y_2]], & [[h(x_{42})]] = j[[y_2]].
\end{array}$$

Now we are ready to calculate the obstruction cocycle $c'_{Q_{4n}}(h)(e)$.

(A) In the case $a_1 = a_2 = a_3 = a_4$, according to previous discussion, $c'_{Q_{4n}}(h)(e) = [[h(x_{31})]] = [[y_2]]$.

(B) The remaining cases of the computation of $[c'_{Q_{4n}}(h)(e)]$ are the following.

(B.1) If $a_2 = a_4$ and $a_1 \neq a_3$, then

$$[c'_{Q_{4n}}(h)(e)] = ((-1)^{(n+1)(a_1+a_2+a_3)}(1+(-1)^{\binom{n}{2}}) + (-1)^{(n+1)a_1}(1+(-1)^{\binom{n}{2}}))[[y_1]] + K.$$

(B.2) If $a_2 = a_4 \neq a_1 = a_3$, then

$$[c(h)(e)] = ((-1)^{(n+1)(a_1+a_2+a_3)} + (-1)^{(n+1)a_1+\binom{n}{2}})[[y_1]] + K.$$

(B.3) If $a_1 = a_3$ and $a_2 \neq a_4$, then

$$[c'_{Q_{4n}}(h)(e)] = ((-1)^{(n+1)(a_1+a_2+a_3)}(1+(-1)^{\binom{n}{2}}) + (-1)^{(n+1)a_1}(1+(-1)^{\binom{n}{2}}))[[y_1]] + K.$$

In particular for $(a_1, a_2, a_3, a_4) = (p, p, p, 2p)$

$$[c'_{Q_{4n}}(h)(e)] = 2((1+(-1)^{\binom{5p}{2}}))[[y_1]] + K.$$

(B.4) If $a_1 = a_2 \neq a_3 = a_4$, then

$$\begin{aligned}
[c'_{Q_{4n}}(h)(e)] = \ & ((-1)^{(n+1)(a_1+a_2+a_3)} + (-1)^{(n+1)a_1+\binom{n}{2}})[[y_1]] - \\
& ((-1)^{(n+1)(a_1+a_2)} + 1)(1+(-1)^{\binom{n}{2}})[[y_2]] + K.
\end{aligned}$$

(B.5) If $a_2 = a_3 \neq a_1 = a_4$, then

$$\begin{aligned}
[c'_{Q_{4n}}(h)(e)] = \ & ((-1)^{(n+1)(a_1+a_2+a_3)} + (-1)^{(n+1)a_1})(1+(-1)^{\binom{n}{2}})[[y_1]] - \\
& (1+(-1)^{(n+1)(a_1+a_2)})[[y_2]] + K.
\end{aligned}$$

(B.6) In all the remaining cases we have

$$\begin{aligned}
[c'_{Q_{4n}}(h)(e)] = \ & ((-1)^{(n+1)(a_1+a_2+a_3)} + (-1)^{(n+1)a_1})(1+(-1)^{\binom{n}{2}})[[y_1]] - \\
& ((-1)^{(n+1)(a_1+a_2)} + 1)(1+(-1)^{\binom{n}{2}})[[y_2]] + K.
\end{aligned}$$

■

4.1.3 The obstruction cocycle $c_{\mathbb{Z}_n}(h)$

In this section we perform similar computation of the obstruction cocycle for a $\mathbb{Z}_n$-map $h : S^3 \to W_n$ in general position, and complete the calculations originally started in [23]. Here the arrangement $\mathcal{A}(\alpha)$ is the minimal $\mathbb{Z}_n$-invariant subspace arrangement generated by $L(\alpha)$ defined by (1).

Theorem 12 *Let $c'_{\mathbb{Z}_n}(h) \in C^3_{\mathbb{Z}_n}(S^3, H_2(M;\mathbb{Z}))$ be the obstruction cocycle for the map h defined above. Let K be the additive subgroup of $H_2(M;\mathbb{Z})$ generated by the elements of the form $g \cdot x - x$, $g \in \mathbb{Z}_n$, $x \in H_2(M;\mathbb{Z})$. Then the group of coinvariants is $H_2(M;\mathbb{Z})_{\mathbb{Z}_n} = H_2(M;\mathbb{Z})/K$.*

(A) Let $(a_1, a_2, a_3, a_4) = (p, p, p, p)$. Then $[c'_{\mathbb{Z}_n}(h)(d_3)] = (-1)^p[[y_1]] + K = [[y_2]] + K$.

(B) Let $(a_1, a_2, a_3, a_4) = (p, q, p, q)$. Then $[c'_{\mathbb{Z}_n}(h)(d_3)] = ((-1)^p + (-1)^q)[[y_1]] + K$.

(C) Let $(a_1, a_2, a_3, a_4) \notin \{(p, p, p, p), (p, q, p, q)\}$. Then

$$[c'_{\mathbb{Z}_n}(h)(d_3)] = ((-1)^{(n+1)a_1} + (-1)^{(n+1)(a_1+a_2+a_3)})[[y_1]] - ((-1)^{(n+1)(a_1+a_2)} + 1)[[y_2]] + K$$

(D) Specially, let $(a_1, a_2, a_3, a_4) = (p, p, p, 2p)$. Then

$$[c'_{\mathbb{Z}_n}(h)(d_3)] = 2[[y_1]] - 2[[y_2]] + K.$$

Proof. The proof goes along the lines of the proof of theorems 4.1 and 4.2 in [23]. We use again the formulas

$$c_{\mathbb{Z}_n}(h)(\theta) = \sum_{y \in h(\theta)\cap D(\alpha)} \mathrm{I}(h(\theta), L_y)[y] \text{ and } c'_{Q_{4n}}(h)(d_3) = \sum_{\theta \subset d_3} \sum_{y \in h(\theta)\cap D(\alpha)} \mathrm{I}(h(\theta), L_y)[y].$$

(A) If $(a_1, a_2, a_3, a_4) = (p, p, p, p)$, then there is only one simplex

$$\epsilon^{-a_1}\theta_1 = \epsilon^{-a_1-a_2}\theta_2 = \epsilon^{-a_1-a_2-a_3}\theta_3 = \theta_4$$

which belongs to the cell d_3 and intersects nontrivially the singular set $h^{-1}(D(\alpha))$. In this case there is only one element of the set $d_3 \cap h^{-1}(D(\alpha))$. Thus

$$c'_{\mathbb{Z}_n}(h)(d_3) = \det(\epsilon^{-a_1})\epsilon^{-a_1}[[y_1]] = (-1)^{(4p+1)p}[[y_1]].$$

(B) If $(a_1, a_2, a_3, a_4) = (p, q, p, q)$, there is only one simplex which belongs to the cell d_3 and nontrivially intersects the singular set $h^{-1}(\cup\mathcal{A}(\alpha))$. Note that in this case $\left|d_3 \cap h^{-1}(D(\alpha))\right| = 2$. This implies that

$$c'_{\mathbb{Z}_n}(h)(d_3) = \det(\epsilon^{-a_1})\epsilon^{-a_1}[[y_1]] + \det(\epsilon^{-a_1-a_2-a_3})\epsilon^{-a_1-a_2-a_3}[[y_3]].$$

(C) In all remaining cases $\left|d_3 \cap h^{-1}(D(\alpha))\right| = 4$ and

$$\begin{aligned} c'_{\mathbb{Z}_n}(h)(d_3) = \; & \det(\epsilon^{-a_1})\epsilon^{-a_1}[[y_1]] - \det(\epsilon^{-a_1-a_2})\epsilon^{-a_1-a_2}[[y_2]] + \\ & \det(\epsilon^{-a_1-a_2-a_3})\epsilon^{-a_1-a_2-a_3}[[y_3]] - [[y_4]]. \end{aligned} \qquad \blacksquare$$

4.2 Homology groups $H_2(M(\alpha);\mathbb{Z})$ as G-modules

Our objective in this section is to compute the torsion subgroups of both $H_2(M;\mathbb{Z})_{\mathbb{Z}_n}$ and $H_2(M;\mathbb{Z})_{Q_{4n}}$ (Proposition 7). In the first case $M = W_n - \cup\mathcal{A}(\alpha)$ is the complement of the minimal $\mathbb{Z}_n$-arrangement $\mathcal{A}(\alpha)$ containing subspace $L(\alpha)$, and in the second case $\mathcal{A}(\alpha)$ is the minimal D_{2n}-arrangement containing $L(\alpha)$ (defined by (1)).

The idea is to use Poincaré-Alexander duality and work with the arrangement $\mathcal{A}(\alpha)$ instead of the complement M. The complement $M = W_n - \cup\mathcal{A}(\alpha)$ can be written in the form $S^{n-1} - \cup\widehat{\mathcal{A}}(\alpha)$, where $\widehat{\mathcal{A}}(\alpha)$ is the compactification of the arrangement $\mathcal{A}(\alpha)$. There is a sequence of isomorphisms

$$\begin{aligned} H_2(M, \mathbb{Z}) &= H_2(S^{n-1} - \cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z}) && \text{Poincaré-Alexander duality} \\ &\cong H^{(n-1)-2-1}(\cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z}) && \text{Universal Coefficient theorem} \\ &\cong \mathrm{Hom}(H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}(H_{n-3}(\cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z}). \end{aligned}$$

The isomorphism of universal coefficient theorem is a $\mathbb{Z}_n$, respectively a Q_{4n}-map,but the Poincaré-Alexander duality map is a $\mathbb{Z}_n$, i.e. Q_{4n}-map up to a orientation character. Let o be an orientation of

the $\mathbb{Z}_n$, respectively Q_{4n}-sphere S^{n-1}. Then o determines Poincaré-Alexander duality map γ_o [15]. In particular, the mapping is Q_{4n}-equivariant up to the orientation $g\cdot o = \det(g)\cdot o$, where $g \in \mathbb{Z}_n \subseteq GL_n(\mathbb{R})$, i.e. $g \in Q_{4n} \subseteq GL_n(\mathbb{R})$. Since the maximal elements of the arrangement $\mathcal{A}(\alpha)$ are $(n-4)$-dimensional linear subspaces, it follows that $H_{n-3}(\cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z}) = 0$ and $\mathrm{Ext}(H_{n-3}(\cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z}) = 0$. Hence,

$$H_2(M(\mathcal{A}(\alpha)), \mathbb{Z}) \cong \mathrm{Hom}(H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z}). \tag{2}$$

The Ziegler-Živaljević formula implies the following decomposition (assuming $\mathbb{Z}$ as coefficients)

$$\begin{aligned} H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha)) &\cong \bigoplus_{p\in P(\alpha)} H_{n-4}(\Delta(P(\alpha)_{<p}) * S^{\dim p}) \cong \bigoplus_{p\in P(\alpha)} H_{n-4}(\Sigma^{\dim p+1}(\Delta(P(\alpha)_{<p}))) \\ &\cong \bigoplus_{p\in P(\alpha)} \tilde{H}_{n-5-\dim p}(\Delta(P(\alpha)_{<p})) \cong \bigoplus_{d=0}^{n-4} \bigoplus_{p\in P(\alpha):\dim p=d} \tilde{H}_{n-5-d}(\Delta(P(\alpha)_{<p})). \end{aligned}$$

where $P(\alpha)$ is the intersection poset of the arrangement $\mathcal{A}(\alpha)$. Thus, in both cases $\mathbb{Z}_n$ and Q_{4n} we have to determine $(\forall p \in P(\alpha))\ \tilde{H}_{n-5-\dim p}(\Delta(P(\alpha)_{<p}))$ with $\tilde{H}_{-1}(\emptyset) = \mathbb{Z}$. Observe that the dimension of the simplicial complex $\Delta(P(\alpha)_{<p})$ is less or equal then $n-5-\dim p$ and $\tilde{H}_{n-5-\dim p}(\Delta(P(\alpha))_{<p}) \neq 0$ implies that there must be at least one chain of length $n-5-\dim p$.

4.2.1 Case 1: $\mathcal{A}(\alpha)$ as the minimal $\mathbb{Z}_n$-invariant subspace arrangement

Theorem 13 *Let $M = W_n - \cup\mathcal{A}(\alpha)$ be the complement of the minimal $\mathbb{Z}_n$-arrangement $\mathcal{A}(\alpha)$ containing subspace $L(\alpha)$ defined by 1. Let $\alpha = (a_1, a_2, a_3, a_4) \in \mathbb{N}^4$ and $a_1 + a_2 + a_3 + a_4 = n$. Then*

(A) If $\alpha = (p,p,p,p)$, then $H_2(M;\mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n/\epsilon^p\mathbb{Z}_n] \cong \mathbb{Z}^{n/4}$;

(B) If $\alpha = (p,q,p,q)$ and $p \neq q$, then $H_2(M;\mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n/\epsilon^{p+q}\mathbb{Z}_n] \cong \mathbb{Z}^{n/2}$;

(C) If $\alpha = (p,p,p,2p)$, then $H_2(M;\mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n] \oplus \frac{n}{5}\mathbb{Z}^4 \cong \mathbb{Z}^{n+\frac{4n}{5}}$;

(D) If $\alpha = (1,1,1,3)$, then $H_2(M;\mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_6] \oplus \mathbb{Z}[\mathbb{Z}_6] \oplus \mathbb{Z} \cong \mathbb{Z}^{13}$;

(E) If $\alpha = (q,q,q,p)$, $p \neq q$, $n > 6$, then $H_2(M;\mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n] \oplus \mathbb{Z}[\mathbb{Z}_n] \cong \mathbb{Z}^{2n}$;

(F) If $\alpha = (p,q,p,p+q)$, $p \neq q$, then $H_2(M;\mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n] \oplus \mathbb{Z}[\mathbb{Z}_n] \cong \mathbb{Z}^{2n}$;

(G) If $\alpha = (p,q,p+q,p+q)$, then $H_2(M;\mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n] \oplus \mathbb{Z}[\mathbb{Z}_n] \oplus \mathbb{Z}[\mathbb{Z}_n/\epsilon^{p+q}\mathbb{Z}_n]$;

(H) In all the remaining cases $H_2(M;\mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n]$.

Before discussing details of the proof let us make a few general observations. The computation of $H_2(M;\mathbb{Z})$ will rely on the isomorphism (2) and the fact (which will be proved) that the homology $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z})$ is free. The computation of the group $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z})$ is based on the Z-Ž formula and again we will have to discuss each of the cases separately. The Z-Ž formula implies that we actually compute $\tilde{H}_{\dim L - \dim p - 1}(\Delta(P(\alpha))_{<p}; \mathbb{Z})$ for each $p \in P(\alpha)$.

Proof. (A) In this case there are $n/4$ maximal elements L, ϵL, ..., $\epsilon^{p-1}L$ of the arrangement $\mathcal{A}(\alpha)$. Since for every $i, j \in \{0, .., p-1\}$, $i \neq j$,

$$\dim L - \dim(\epsilon^i L \cap \epsilon^j L) > 1$$

the Z-Ž formula implies that only maximal elements contribute to $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z})$.

(B) and **(H)** The argument for these cases is exactly the same as for (A), except for the number of maximal elements of the arrangement $\mathcal{A}(\alpha)$.

(C) There are four maximal elements $\epsilon^p L$, $\epsilon^{2p} L$, $\epsilon^{3p} L$ and $\epsilon^{4p} L$ with the $\dim(L \cap \epsilon^{k\cdot p} L) = \dim L - 1$ for $k = 1, .., 4$. In addition

$$L \cap \epsilon^p L = L \cap \epsilon^{2p} L = L \cap \epsilon^{3p} L = L \cap \epsilon^{4p} L.$$

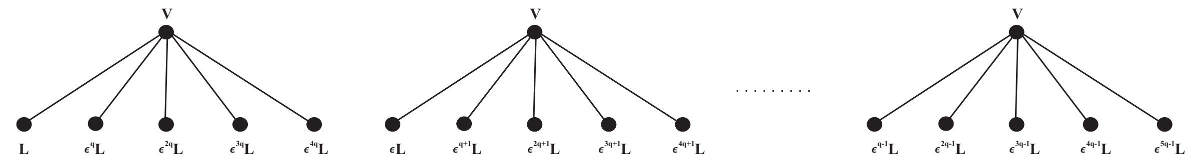

Figure 1: The Hasse diagram for the case (C)

Let $V = L \cap \epsilon^q L \cap \epsilon^{2q} L \cap \epsilon^{3q} L \cap \epsilon^{4q} L$. Then the Hasse diagram of the subposet $\{p \in P(\alpha) : \dim p \geq n-5\}$ is depicted in the Figure 1.

Before using the Z-Ž decomposition observe that for $n \geq 5$

$$(\forall p \in P(\alpha)) \dim p \leq n-6 \implies \tilde{H}_{n-5-\dim p}(\Delta(P(\alpha)_{<p})) = 0.$$

Indeed, for each element $q \in P(\alpha)_{<p}$ such that $\dim q = n-4$ there exists a unique element $p_q \in P(\alpha)_{<p}$ with the property $\dim p_q = n-5$ and $q < p_q$. There is a monotone map $f : P(\alpha)_{<p} \to P(\alpha)_{<p} - \{q | \dim q = n-4\}$ defined by

$$q \quad \longmapsto \quad \begin{cases} q, & \text{for } \dim q \leq n-5 \\ p_q & \text{for } \dim q = n-4 \end{cases}$$

which satisfies conditions of the Quillen fiber lemma. This implies that f induces a homotopy equivalence, hence

$$\tilde{H}_{n-5-\dim p}(\Delta(P(\alpha)_{<p})) = \tilde{H}_{n-5-\dim p}(\Delta(Q)) = 0$$

since $\dim \Delta(Q) < n-5-\dim p$.

Thus, the only relevant part of the poset $P(\alpha)$ for the computation of $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z})$ is the part in the above picture.

(D) The proof follows from the Z-Ž decomposition and the Hasse diagram of the intersection poset, shown in the Figure 2.

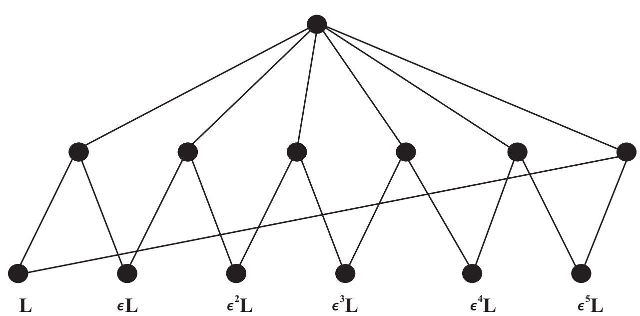

Figure 2: The Hasse diagram for the case (D)

(E) When $\alpha = (a_1, a_2, a_3, a_4) = (q, q, q, p)$, $p \neq 2q$, there are exactly two maximal elements $\epsilon^q L$ and $\epsilon^{2q+p} L$ in $P(\alpha)$ with the maximal L intersection, $\dim(L \cap \epsilon^q L) = \dim(L \cap \epsilon^{2q+p} L) = \dim L - 1$. It is not hard to describe the first three levels of the intersection poset $P(\alpha)$. There are n elements $L, \epsilon L, .., \epsilon^{n-1} L$ of dimension $n-4$, n elements

$$L \cap \epsilon^q L, \epsilon^q L \cap \epsilon^{2q} L, ..., \epsilon^{2q+p} L \cap L;\ \epsilon L \cap \epsilon^{q+1} L, \epsilon^{q+1} L \cap \epsilon^{2q+1} L, ..., \epsilon^{2q+p+1} L \cap \epsilon L;$$
$$\epsilon^{d-1} L \cap \epsilon^{q+d-1} L, \epsilon^{q+d-1} L \cap \epsilon^{2q+d-1} L, ..., \epsilon^{2q+p+d-1} L \cap \epsilon^{d-1} L$$

of dimension $n-5$ (where $d = (q, n)$) and finally n elements

$$L \cap \epsilon^q L \cap \epsilon^{2q+p} L, \epsilon(L \cap \epsilon^q L \cap \epsilon^{2q+p} L), .., \epsilon^{n-1}(L \cap \epsilon^q L \cap \epsilon^{2q+p} L)$$

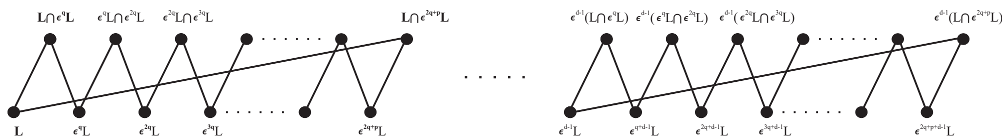

Figure 3: The first Hasse diagram for the case (E)

of dimension $n-6$. Thus, the Hasse diagram of the subposet $\{p \in P(\alpha) : \dim p \geq n-5\}$ is of the form depicted in the Figure 3.

We prove that only this part of the Hasse diagram of the intersection poset $P(\alpha)$ contributes to the homology $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z})$. Indeed, for $n > 6$

$$(\forall p \in P(\alpha)) \dim p \leq n-6 \implies \tilde{H}_{n-5-\dim p}(\Delta(P(\alpha)_{<p})) = 0.$$

Observe that $\dim \Delta(P(\alpha)_{<p}) \leq n-5-\dim p$. If $\dim \Delta(P(\alpha)_{<p}) < n-5-\dim p$ we have nothing to do, so we assume that $\dim \Delta(P(\alpha)_{<p}) = n-5-\dim p$. Let $K = \{q \in P(\alpha)_{<p} | \dim q \neq n-6\}$. Let us show that the inclusion $i : K \hookrightarrow P(\alpha)_{<p}$ satisfies the assumptions of the Quillen fiber lemma. It suffices to check that for every $q = \epsilon^i(L \cap \epsilon^q L \cap \epsilon^{2q+p} L) \in P(\alpha)_{<p} - K$ the order complex $\Delta(i^{-1}((P(\alpha)_{<p})_{\leq q}))$ is contractible. For such a $q = \epsilon^i(L \cap \epsilon^q L \cap \epsilon^{2q+p} L)$, the Hasse diagram of the poset $i^{-1}((P(\alpha)_{<p})_{\leq q})$ is depicted in the Figure 4 we observe that $\Delta(i^{-1}((P(\alpha)_{<p})_{\leq q}))$ is obviously contractible. Hence, $\Delta(P(\alpha)_{<p}) \simeq \Delta(K)$. Since, $\dim \Delta(P(\alpha)_{<p}) < n-5-\dim p$, we conclude that $\tilde{H}_{n-5-\dim p}(K) = 0$.

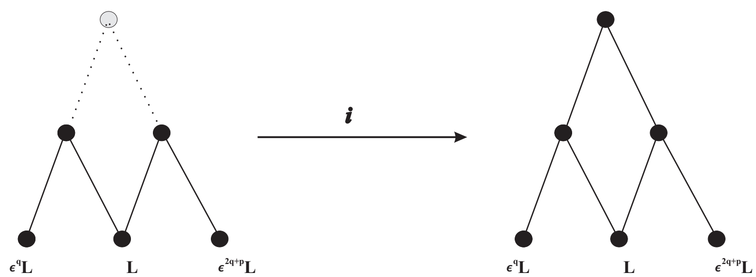

Figure 4: The second Hasse diagram for the case (E)

As before, from the Ziegler-Živaljević formula we deduce the following decomposition

$$H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha); \mathbb{Z}) \cong \bigoplus_{d=n-5}^{n-4} \bigoplus_{p \in P(\alpha) : \dim p = d} \tilde{H}_{n-5-d}(\Delta(P(\alpha)_{<p}); \mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n] \oplus \mathbb{Z}[\mathbb{Z}_n].$$

(F) There are exactly two maximal elements $\epsilon^{p+q}L$ and $\epsilon^{2p+q}L$ with the property

$$\dim(L \cap \epsilon^{p+q}L) = \dim(L \cap \epsilon^{2p+q}L) = \dim L - 1.$$

Since the subposet $\{p \in P(\alpha) : \dim p \geq n-5\}$ of $P(\alpha)$ has the same shape as the appropriate subposet in the proof of the part (E), it is not hard to see that the computation for this case goes completely along the lines of the preceding proof.

(G) In this case there are two maximal elements $\epsilon^{p+q}L$ and $\epsilon^{2p+2q}L$ with the maximal L intersection , i.e.

$$\dim(L \cap \epsilon^{p+q}L) = \dim(L \cap \epsilon^{2p+2q}L) = \dim L - 1.$$

The elements of $P(\alpha)$ of dimension $n-5$ are: $L \cap \epsilon^{p+q}L, \epsilon(L \cap \epsilon^{p+q}L), .., \epsilon^{n-1}(L \cap \epsilon^{p+q}L)$. There are $p+q$ elements of dimension $n-6$ for $n > 6$: $L \cap \epsilon^{p+q}L \cap \epsilon^{2p+2q}L$, $\epsilon(L \cap \epsilon^{p+q}L \cap \epsilon^{2p+2q}L), .., \epsilon^{p+q-1}(L \cap \epsilon^{p+q}L \cap \epsilon^{2p+2q}L)$ and only one for $n = 6$. We use this information to draw the Hasse diagram (Figure 5) for the subposet $\{p \in P(\alpha) : \dim p \geq n-6\}$.

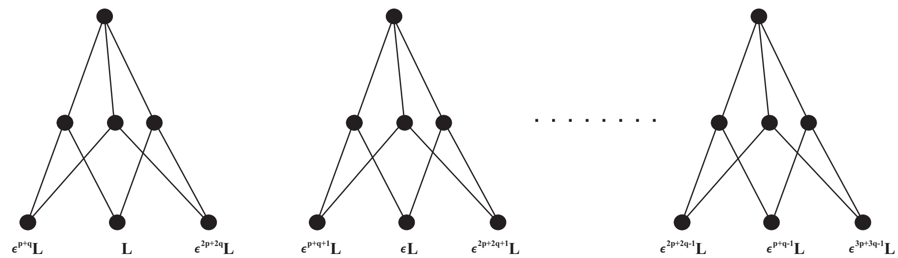

Figure 5: The Hasse diagram for the case (G)

Let us convince ourselves that for $n > 6$

$$(\forall p \in P(\alpha)) \dim p \leq n-7 \Longrightarrow \tilde{H}_{n-5-\dim p}(\Delta(P(\alpha)_{<p})) = 0.$$

Let $R \stackrel{def}{=} P(\alpha)_{<p} \cap \{p \in P(\alpha) | \dim p = n-6\} = \{p_1, .., p_k\}$ where $k \leq p+q$ and $Q = P(\alpha)_{<p} \cap \{q \in P(\alpha) | \dim q \leq n-6\}$. Since for every element $q \in P(\alpha)_{<p} \cap \{p \in P(\alpha) | \dim p > n-6\}$ there exists an unique element $p_q \in R$ such that $q < p_q$, we can define a monotone map $f : P(\alpha)_{<p} \to Q$ by the formula

$$f(q) = \begin{cases} q & , \dim q \leq n-6 \\ p_q & , \dim q > n-6 \end{cases}$$

Again the Quillen fiber lemma implies that f induces a homotopy equivalence $\Delta(P(\alpha)_{<p}) \simeq \Delta(Q)$. Indeed, for each $q \in Q$ the poset $f^{-1}((P(\alpha)_{<p})_{\leq q})$ has a maximum, hence $\Delta(f^{-1}((P(\alpha)_{<p})_{\leq q}))$ is contractible. It follows from $\dim \Delta(Q) < n-5-\dim p$ that

$$\tilde{H}_{n-5-\dim p}(\Delta(P(\alpha)_{<p})) = \tilde{H}_{n-5-\dim p}(\Delta(Q)) = 0.$$

From here we infer the that Ziegler-Živaljević formula decomposition reduces to the first three levels of the Hasse diagram of $P(\alpha)$ and

$$H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n] \oplus \mathbb{Z}[\mathbb{Z}_n] \oplus \mathbb{Z}[\mathbb{Z}_n/\epsilon^{p+q}\mathbb{Z}_n].$$

(H) In all the remaining cases there are exactly n maximal elements of the arrangement, an no intersection of two maximal elements is of dimension $n-5$. Therefore, only maximal elements contribute to the homology $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha), \mathbb{Z})$. ∎

4.2.2 Case 2: $\mathcal{A}(\alpha)$ as the minimal Q_{4n}-invariant subspace arrangement

Theorem 14 *Let $M = W_n - \cup\mathcal{A}(\alpha)$ be the complement of the minimal D_{2n}-invariant arrangement $\mathcal{A}(\alpha)$ containing the subspace $L(\alpha)$ defined by (1) (or equivalently the Q_{4n}-arrangement, where j^2 acts trivially). Let $\alpha = (a_1, a_2, a_3, a_4) \in \mathbb{N}^4$ and $a_1 + a_2 + a_3 + a_4 = n$. Then*

(A) If $\alpha = (p, p, p, p)$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n/\epsilon^p\mathbb{Z}_n] \cong \mathbb{Z}^{n/4}$;

(B) If $\alpha = (p, q, p, q)$ and $p \neq q$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n/\epsilon^{p+q}\mathbb{Z}_n] \cong \mathbb{Z}^{n/2}$;

(C) If $\alpha = (p, p, p, 2p)$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n] \oplus \frac{n}{5}\mathbb{Z}^4 \cong \mathbb{Z}^{n+\frac{4n}{5}}$;

(D) If $\alpha = (1, 1, 1, 3)$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_6] \oplus \mathbb{Z}[\mathbb{Z}_6] \oplus \mathbb{Z} \cong \mathbb{Z}^{13}$;

(E) If $\alpha = (q, q, q, p)$ and $p \notin \{q, 2q\}$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n] \oplus \mathbb{Z}[\mathbb{Z}_n] \cong \mathbb{Z}^{2n}$;

(F) If $\alpha = (p, q, p, p+q)$ and $p \neq q$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n] \oplus \mathbb{Z}[\mathbb{Z}_n] \cong \mathbb{Z}^{2n}$;

(G) If $\alpha = (p, p, 2p, 2p)$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n] \oplus \mathbb{Z}[\mathbb{Z}_n] \oplus \mathbb{Z}[\mathbb{Z}_n/\epsilon^{2p}\mathbb{Z}_n]$;

(H) If $\alpha = (p, q, p+q, p+q)$ and $p \neq q$, then $H_2(M; \mathbb{Z}) \cong ?$.

(I) If $\alpha = (p,p,q,q)$ and $p \neq q$, $p \neq 2q$, $q \neq 2p$, then $H_2(M;\mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n]$;

(J) If $\alpha = (p,q,r,q)$ and $r \notin \{q,p,p+q\}$, then $H_2(M;\mathbb{Z}) \cong \mathbb{Z}[\mathbb{Z}_n]$;

(K) If $\alpha = (p,q,r,p+q)$ and $r \notin \{p,p+q\}$, then $H_2(M;\mathbb{Z}) \cong ?$;

(L) If $\alpha = (p,p,q,q+r)$, and $p \neq q$, $r \neq 0$, then $H_2(M;\mathbb{Z}) \cong ?$;

(M) In all the remaining cases $H_2(M;\mathbb{Z}) \cong \mathbb{Z}[D_{2n}]$.

Remark 15 *The computation of $H_2(M;\mathbb{Z})$ in the cases (H), (K) and (L) is probably within reach of the existing methods, however we leave it as an open problem.*

Like in the preceding section the computation of $H_2(M;\mathbb{Z})$ will rely on the isomorphism (2) and the fact that the homology $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ is free. This means that we actually compute $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$, i.e. with the help of the Z-Ž formula we compute $\tilde{H}_{\dim L - \dim p - 1}(\Delta(P(\alpha))_{<p};\mathbb{Z})$ for each $p \in P(\alpha)$.

Proof. (A)-(G) and (I)-(J) In these cases minimal D_{2n}-arrangement containing the subspace $L(\alpha)$ coincides with the minimal $\mathbb{Z}_n$-arrangement containing the same subspace $L(\alpha)$. Thus the assertions are consequences of Theorem 13.

(M) In all the remaining cases there are exactly $2n$ maximal elements of the arrangement, an no intersection of two maximal elements is of dimension $n-5$. Therefore, only maximal elements contribute to the homology $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$. ■

4.3 The coinvariants of $H_2(M(\alpha);\mathbb{Z})$

Relying on the results from the previous section we are able to read off the $\mathbb{Z}_n$, respectively Q_{4n}, coinvariants of the module $H_2(M;\mathbb{Z})$.

4.3.1 Case 1: $\mathbb{Z}_n$-coinvariants of $H_2(M;\mathbb{Z})$

Theorem 16 *Let $M = W_n - \cup\mathcal{A}(\alpha)$ be the complement of the minimal $\mathbb{Z}_n$-arrangement $\mathcal{A}(\alpha)$ containing the subspace $L(\alpha)$ defined by 1. Let $\alpha = (a_1,a_2,a_3,a_4) \in \mathbb{N}^4$ and $a_1+a_2+a_3+a_4 = n$. Then*

	α	$H_2(M;\mathbb{Z})_{\mathbb{Z}_n}$
(A)	(p,p,p,p)	$\mathbb{Z}_2$
(B)	(p,q,p,q), $p \neq q$	$\mathbb{Z}$
(C)	$(p,p,p,2p)$	$\mathbb{Z} \oplus \mathbb{Z}_5$
(D)	$(1,1,1,3)$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$
(E)	(q,q,q,p), $p \notin \{q,2q\}$	$\mathbb{Z} \oplus \mathbb{Z}$
(F)	$(p,q,p,p+q)$, $p \neq q$	$\mathbb{Z} \oplus \mathbb{Z}$
(G)	$(p,q,p+q,p+q)$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
(H)	*In all the remaining cases*	$\mathbb{Z}$

To prove this theorem we have to keep in mind that the of Poincaré-Alexander duality map is an equivariant up to an orientation character. To compute $\mathbb{Z}_n$-coinvariants $H_2(M(\mathcal{A}(\alpha)),\mathbb{Z})_{\mathbb{Z}_n}$ we pass to the $\mathbb{Z}_n$-module $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$, but with the modified action. More precisely, let $l \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ and $g \in \mathbb{Z}_n$, then

$$g \cdot_{\mathrm{m}} l = \det(g)\ g \cdot l$$

where $\cdot_{\mathrm{m}}$ is the new modified action, and $\cdot$ the old action. Let us denote by $\sim$ the equivalence relation on $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ such that the subgroup of elements equivalent to zero is generated by the elements of the form $g \cdot_{\mathrm{m}} x - x$, $g \in \mathbb{Z}_n$, $x \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$. In other words there is an isomorphism $H_2(M;\mathbb{Z})_{\mathbb{Z}_n} \cong H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})/\sim$.

Proof. **(A)** Let $\alpha = (p,p,p,p)$ and $l \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be the homology class induced by the subspaces L. The equality $L = \epsilon^p \cdot L$ translates to the equality $l = (-1)^{p-1}\epsilon^p \cdot l$ in homology, because $\det(\epsilon^p|_L) = (-1)^{p-1}$. Now in coinvariants we have

$$l \sim \epsilon^p \cdot_{\mathrm{m}} l = \det(\epsilon^p)\ \epsilon^p \cdot l = (-1)^{p(n+1)}\ \epsilon^p \cdot l = (-1)^{p(4p+1)}(-1)^{p-1}l = -l$$

which implies the first result.

(B) For $\alpha = (p,q,p,q)$ let l be the homology class induced by the subspaces L. The equality $L = \epsilon^{p+q} \cdot L$ becomes relation $l = (-1)^{p+q}\epsilon^{p+q} \cdot l$ in homology ($\det(\epsilon^p|_L) = (-1)^{p+q}$), and in coinvariants

$$l \sim \epsilon^{p+q} \cdot_{\mathrm{m}} l = \det(\epsilon^{p+q})\ \epsilon^{p+q} \cdot l = (-1)^{(p+q)(n+1)}\ \epsilon^{p+q} \cdot l = (-1)^{(p+q)(2p+2q+1)}(-1)^{p+q}l = l.$$

(C) Let $l,k \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be homology classes induced by subspaces L and $L\cap\epsilon^p L$. Then l generates $\mathbb{Z}[\mathbb{Z}_n]$ part in $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ and there are no nontrivial relations on l. For k there is only one relevant relation. The homology equality $k + \epsilon^p \cdot k + \epsilon^{2p} \cdot k + \epsilon^{3p} \cdot k + \epsilon^{4p} \cdot k = 0$ implies the relation

$$k + (-1)^{2(p(5p+1))}\epsilon^p \cdot k + (-1)^{2(2p(5p+1))}\epsilon^{2p} \cdot k + (-1)^{2(3p(5p+1))}\epsilon^{3p} \cdot k + (-1)^{2(4p(5p+1))}\epsilon^{4p} \cdot k = 0$$

or equivalently

$$k + (-1)^{p(5p+1)}\epsilon^p \cdot_{\mathrm{m}} k + (-1)^{2p(5p+1)}\epsilon^{2p} \cdot_{\mathrm{m}} k + (-1)^{3p(5p+1)}\epsilon^{3p} \cdot_{\mathrm{m}} k + (-1)^{4p(5p+1)}\epsilon^{4p} \cdot_{\mathrm{m}} k = 0.$$

Passing to coinvariants we have

$$k + (-1)^{p(5p+1)}k + k + (-1)^{3p(5p+1)}k + k \sim 0 \iff 5k \sim 0.$$

(D) Let $l,k,h \in H_2(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be homology classes induced by subspaces L, $L\cap\epsilon L$ and $\bigcap_{i=0}^{5}\epsilon^i L$. Then l generates the first copy of $\mathbb{Z}[\mathbb{Z}_6]$ and k the second copy of $\mathbb{Z}[\mathbb{Z}_6]$ in the $\mathbb{Z}_6$-module $H_2(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$. There are no relations on l and k, so the classes of l and k generate the first and the second copy of $\mathbb{Z}$ in coinvariants. The equality $\bigcap_{i=0}^{5}\epsilon^i L = \epsilon \cdot (\bigcap_{i=0}^{5}\epsilon^i L)$ implies the homology equality $h = \epsilon \cdot h$. Thus we have the relation

$$h \sim \epsilon \cdot_{\mathrm{m}} h = (-1)^7\epsilon \cdot h = -h$$

which produces the $\mathbb{Z}_2$ in coinvariants.

(E) Let $l,k \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be homology classes induced by subspaces L, $L\cap\epsilon^q L$. There are no non-trivial relations on l and k, so the coinvariants are $\mathbb{Z}\oplus\mathbb{Z}$.

(F) Let $l,k \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be homology classes induced by subspaces L, $L\cap\epsilon^{p+q}L$. Again, there are no relations on l and k and the result follows directly.

(G) Let $l,k,h \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be homology classes induced by subspaces L, $L\cap\epsilon^{p+q}L$ and $L\cap\epsilon^{p+q}L\cap\epsilon^{2p+2q}L$. There is only one relevant equality $L\cap\epsilon^{p+q}L\cap\epsilon^{2p+2q}L = \epsilon^{p+q}\cdot(L\cap\epsilon^{p+q}L\cap\epsilon^{2p+2q}L)$ which produces the equality $h = (-1)^{(p+q)(n+1)}\epsilon^{p+q}\cdot h$ in homology (because $\det(\epsilon^{p+q}|_{L\cap\epsilon^{p+q}L\cap\epsilon^{2p+2q}L}) = (-1)^{(p+q)(n+1)}$). These relations become trivial in coinvariants

$$h \sim \epsilon^{p+q} \cdot_{\mathrm{m}} h = (-1)^{(p+q)(n+1)}\epsilon^{p+q}\cdot h = h.$$

(H) Let $l \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be the homology class induced by subspaces L. Then $l, \epsilon\cdot l, ..., \epsilon^{n-1}\cdot l$ are generators of $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ and there are no non trivial relations on these generators. ■

4.3.2 Case 2: Q_{4n}-coinvariants of $H_2(M;\mathbb{Z})$

Theorem 17 *Let $M = W_n - \cup\mathcal{A}(\alpha)$ be the complement of the minimal D_{2n}-arrangement $\mathcal{A}(\alpha)$ containing the subspace $L(\alpha)$ defined by (1) (or Q_{4n}-arrangement, where j^2 acts trivially). Let $\alpha = (a_1,a_2,a_3,a_4) \in \mathbb{N}^4$ and $a_1+a_2+a_3+a_4 = n$. Then*

	α	$H_2(M;\mathbb{Z})_{Q_{4n}}$
(A)	(p,p,p,p)	$\mathbb{Z}_2$
(B)	$(p,q,p,q),\ p\neq q$	$\mathbb{Z}_2$
(C)	$(p,p,p,2p)$	$\mathbb{Z}_2\oplus\mathbb{Z}_5$
(D)	$(1,1,1,3)$	$\mathbb{Z}_2\oplus\mathbb{Z}\oplus\mathbb{Z}_2$
(E)	$(q,q,q,p),\ p\notin\{q,2q\}$	$\mathbb{Z}_2\oplus\mathbb{Z}$
(F)	$(p,q,p,p+q),\ p\neq q$	$\mathbb{Z}_2\oplus\mathbb{Z}$
(G)	$(p,p,2p,2p)$	$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2$
(H)	$(p,q,p+q,p+q),\ p\neq q$	?
(I)	$(p,p,q,q),\ p\neq q,\ p\neq 2q,\ q\neq 2p$	$\mathbb{Z}$
(J)	$(p,q,r,q),\ r\notin\{q,p,p+q\}$	$\mathbb{Z}_2$
(K)	$(p,q,r,p+q),\ r\notin\{p,p+q\}$	?
(L)	$(p,p,q,q+r),\ p\neq q,\ r\neq 0$	?
(M)	*In all the remaining cases*	$\mathbb{Z}$

The proof goes along the lines of the proof of theorem 16. Again, to compute the group $H_2(M;\mathbb{Z})_{Q_n}$ of Q_{4n}-coinvariants, we use the Q_n-module $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$, but with the modified action. Precisely, for $l\in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ and $g\in Q_n$,

$$g\cdot_{\mathrm{m}} l=\det(g)\ g\cdot l.$$

We restrict ourselves to pointing to those relations on generators of $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ which produce non-trivial identities in coinvariants. Again, let $\sim$ be the equivalence relation on $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ with the zero equivalence class generated by elements of the form $g\cdot_{\mathrm{m}} x-x$, $g\in Q_{4n}$, $x\in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$. Thus $H_2(M;\mathbb{Z})_{Q_{4n}}\cong H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})/\sim$.

Proof. **(A)** Since $H_2(M;\mathbb{Z})_{Q_{4n}}$ is a quotient of $H_2(M;\mathbb{Z})_{\mathbb{Z}_n}$, the result follows.

(B) For $\alpha=(p,q,p,q)$, let $l\in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be determined by the subspace L.There are two equalities: $L=\epsilon^{p+q}\cdot L$ and $L=\epsilon^{-q}j\cdot L$, which in homology become equalities $l=(-1)^{p+q}\epsilon^{p+q}\cdot l$ and $l=(-1)^{p+2q+1}\epsilon^{-q}j\cdot l$. These equalities imply that

$$l\sim\epsilon^{p+q}\cdot_{\mathrm{m}} l=\det(\epsilon^{p+q})\ \epsilon^{p+q}\cdot l=(-1)^{(p+q)(n+1)}\ \epsilon^{p+q}\cdot l=(-1)^{(p+q)(2p+2q+1)}(-1)^{p+q}l=l,$$

and

$$l\sim\epsilon^{-q}j\cdot_{\mathrm{m}} l=\det(\epsilon^{-q}j)\ \epsilon^{-q}j\cdot l=(-1)^{(n+1)q+\binom{n}{2}}\ \epsilon^{-q}j\cdot l=(-1)^{p+2q}(-1)^{p+2q+1}l=-l.$$

(C) Let $l,k\in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be homology classes induced by subspaces L and $L\cap\epsilon^pL$. In contrast to the proof of theorem 16 (C), we have a relation on l, which is a consequence of the equality $L=\epsilon^{-2p}j\cdot L$. Thus in homology $l=(-1)^{\binom{5p}{2}+1}\epsilon^{-2p}j\cdot l$ and consequently in coinvariants

$$l\sim\epsilon^{-2p}j\cdot_{\mathrm{m}} l=(-1)^{2p(5p+1)+\binom{5p}{2}}\epsilon^{-2p}j\cdot l=(-1)^{\binom{5p}{2}}\epsilon^{-2p}j\cdot l=-l.$$

There is also one more relation on k which is the consequence of the equality $L\cap\epsilon^pL=\epsilon^{-p}j\cdot(L\cap\epsilon^pL)$. Passing to homology we get the equality $k=(-1)^{p(n+1)+\binom{n}{2}}\epsilon^{-p}j\cdot k$. In coinvariants we get the trivial relation

$$k\sim\epsilon^{-p}j\cdot_{\mathrm{m}} k=(-1)^{p(n+1)+\binom{n}{2}}\epsilon^{-p}j\cdot k=k$$

and the second summand is just $\mathbb{Z}_5$.

(D) Let $l,k,h\in H_2(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be induced by subspaces L, $L\cap\epsilon L$ and $\bigcap_{i=0}^5\epsilon^iL$. The equalities $L=\epsilon^{-3}j\cdot L$ and $L\cap\epsilon L=\epsilon^{-2}j\cdot(L\cap\epsilon L)$ imply the equalities $l=-\epsilon^{-3}j\cdot l$ and $k=-\epsilon^{-2}j\cdot k$ (since $\det(\epsilon^{-3}j|_L)=-1$ and $\det(\epsilon^{-2}j|_{L\cap\epsilon L})=-1$). We do not consider equality $\bigcap_{i=0}^5\epsilon^iL=j(\bigcap_{i=0}^5\epsilon^iL)$, because $\bigcap_{i=0}^5\epsilon^iL=\epsilon(\bigcap_{i=0}^5\epsilon^iL)$ already produced $\mathbb{Z}_2$-torsion. Thus in coinvariants

$$l\sim\epsilon^{-3}j\cdot_{\mathrm{m}} l=\epsilon^{-3}j\cdot l=-l\ \text{ and }k\sim\epsilon^{-2}j\cdot_{\mathrm{m}} k=-\epsilon^{-2}j\cdot k=k.$$

(E) Let $l,k \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be homology classes induced by subspaces L, $L\cap\epsilon^q L$. Now we have two equalities $L = \epsilon^{-p}j\cdot L$ and $L\cap\epsilon^q L = \epsilon^{-(2p+2q)}j\cdot(L\cap\epsilon^q L)$ in contrast to the similar case in the proof of the theorem 16, (E).These equalities produce equalities $l = (-1)^{(n+1)3q+\binom{n}{2}+1}\epsilon^{-p}j\cdot l$ and $k = (-1)^{(n+1)(q-p)+\binom{n}{2}}\epsilon^{q-p}j\cdot k$ in homology. Passing to coinvariants we have

$$l \sim \epsilon^{-p}j\cdot_{\mathrm{m}} l = (-1)^{(n+1)3q+\binom{n}{2}}\epsilon^{-p}j\cdot l = -l \text{ and } k\sim\epsilon^{q-p}j\cdot_{\mathrm{m}} k = (-1)^{(n+1)(q-p)+\binom{n}{2}}\epsilon^{q-p}j\cdot k = k.$$

(F) Let $l,k \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be homology classes induced by subspaces L, $L\cap\epsilon^{p+q}L$. The equalities $L = \epsilon^{-(p+q)}j\cdot L$ and $L\cap\epsilon^{p+q}L = j\cdot(L\cap\epsilon^{p+q}L)$ imply $l = (-1)^{(n+1)(2p+q)+\binom{n}{2}+1}\epsilon^{-(p+q)}j\cdot l$ and $k = (-1)^{\binom{n}{2}}j\cdot k$ in homology. Therefore, in coinvariants

$$l\sim\epsilon^{-(p+q)}j\cdot_{\mathrm{m}} l = (-1)^{(n+1)(2p+q)+\binom{n}{2}}\epsilon^{-(p+q)}j\cdot l = -l \quad\text{and}\quad k\sim j\cdot_{\mathrm{m}} k = (-1)^{\binom{n}{2}}j\cdot k = k.$$

(G) Let $l,k,h \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be homology classes induced by subspaces L, $L\cap\epsilon^{4p}L$ and $L\cap\epsilon^{2p}L\cap\epsilon^{4p}L$. The relevant equalities are

$$\epsilon^{2p}j\cdot L = L,\ j\cdot(L\cap\epsilon^{4p}L) = L\cap\epsilon^{4p}L,\ j\cdot(L\cap\epsilon^{2p}L\cap\epsilon^{4p}L) = L\cap\epsilon^{2p}L\cap\epsilon^{4p}L$$

which in homology imply

$$l = (-1)^{\binom{n}{2}}\epsilon^{2p}j\cdot l,\ k = (-1)^{\binom{n}{2}}j\cdot k,\ h = (-1)^{\binom{n}{2}+1}j\cdot h.$$

So in coinvariants we have the following relations

$$l\sim\epsilon^{2p}j\cdot_{\mathrm{m}} l = (-1)^{\binom{n}{2}}\epsilon^{2p}j\cdot l = l,\ k\sim j\cdot_{\mathrm{m}} k = k = (-1)^{\binom{n}{2}}j\cdot k = k,\ h\sim j\cdot_{\mathrm{m}} h = (-1)^{\binom{n}{2}}j\cdot h = -h.$$

(I) Let $l \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be the homology class induced by the subspace L. There is only one equality, $L = \epsilon^{-2q}j\cdot L$ which in homology reads as $l = (-1)^{\binom{n}{2}}\epsilon^{-2q}j\cdot l$ and in coinvariants

$$l\sim\epsilon^{-2q}j\cdot_{\mathrm{m}} l = (-1)^{\binom{n}{2}}\epsilon^{-2q}j\cdot l = l$$

(J) Let $l \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ be induced by the subspace L. The equality $\epsilon^p j\cdot L = L$ implies equality $l = (-1)^{p(n+1)+\binom{n}{2}+1}\epsilon^p j\cdot l$ in homology. As before, in coinvariants this leads to the equality

$$l\sim\epsilon^p j\cdot_{\mathrm{m}} l = (-1)^{p(n+1)+\binom{n}{2}}\epsilon^p j\cdot l = -l.$$

(M) In all the remaining cases there are no relations involving the element $l \in H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ induced by subspace L. Since the homology group $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})$ is freely generated by the orbit of l, the group $H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z})_{Q_{4n}}$ of coinvariants is $\mathbb{Z}$. ■

4.4 The main theorem

This is the central section of the paper. We gather together all the information collected in previous sections and use it to obtain a reasonably complete answer to the Problem 3.

Theorem 18 *There does not exist a $\mathbb{Z}_n$-map $F : S^3 \to M(\alpha)$ if and only if*

$$(\exists\ p\in\mathbb{N})\ \alpha = (p,p,p,p) \text{ or } \alpha = (p,p,p,2p).$$

Proof. (A) We first deal with the existence of a $\mathbb{Z}_n$-map. Assume that α is not of the form (p,p,p,p) or $(p,p,p,2p)$. Then the cohomology class of the obstruction cocycle $[c'_{\mathbb{Z}_n}(h)(d_3)]$ is divisible by 2 (Theorem 12). On the other hand this cohomology class is a torsion element of $H_2(M;\mathbb{Z})_{\mathbb{Z}_n}$ (Proposition 7). Since

Theorem 16 implies that the only torsion summands which can appear are copies of $\mathbb{Z}_2$, we conclude that $[c'_{\mathbb{Z}_n}(h)(d_3)] = 0$. Thus a $\mathbb{Z}_n$-map $F : S^3 \to M(\alpha)$ must exists.

(B) Let $\alpha = (p,p,p,p)$. In order to prove that $[c'_{\mathbb{Z}_n}(h)(d_3)] = (-1)^p[[y_1]] + K \neq 0$ we again rely on the isomorphism $\varphi : H_2(M(\mathcal{A}(\alpha)),\mathbb{Z}) \to \mathrm{Hom}(H_{n-4}(\cup\widehat{\mathcal{A}}(\alpha),\mathbb{Z}),\mathbb{Z})$. Using the fact that this isomorphism is related to the linking number, we have that

$$[[y_1]] \to \sum_{i=0}^{p-1} \mathrm{link}(\widehat{\epsilon^i L}, (\widehat{y_1 + L^\perp}))\epsilon^i l = l$$

since $\widehat{L}$ links only with $(\widehat{y_1 + L^\perp})$. Here $\widehat{\epsilon^i L}$ and $(\widehat{y_1 + L^\perp})$ are respectively one-point compactifications of the linear subspace $\epsilon^i L$ and the affine subspace $y_1 + L^\perp$ in the sphere $\widehat{W_n} \approx S^{n-1}$. Now let K be the additive subgroup of $H_2(M;\mathbb{Z})$ generated by the elements of the form $g \cdot x - x$, $g \in \mathbb{Z}_n$, $x \in H_2(M;\mathbb{Z})$. Then the group of coinvariants is $H_2(M;\mathbb{Z})_{\mathbb{Z}_n} = H_2(M;\mathbb{Z})/K$. Since $[c'_{\mathbb{Z}_n}(h)(d_3)] = (-1)^p[[y_1]] + K$ we have that $[c'_{\mathbb{Z}_n}(h)(d_3)]$ is the generator in $H_2(M;\mathbb{Z})_{\mathbb{Z}_n} \cong \mathbb{Z}_2$. We conclude that there does not exist a $\mathbb{Z}_n$-map $F : S^3 \to M(\alpha)$.

(C) Let $\alpha = (p,p,p,2p)$. Like in the preceding case we look at the φ image of the point classes $[[y_1]]$ and $[[y_2]]$,

$$\begin{aligned}
[[y_1]] &\to \sum_{i=0}^{5p-1} \mathrm{link}(\widehat{\epsilon^i L}, (\widehat{y_1 + L^\perp}))\epsilon^i l + \sum_{i=0}^{4p-1} \mathrm{link}(\epsilon^i(\widehat{L \cap \epsilon^p L}), (\widehat{y_1 + L^\perp}))\epsilon^i k = l + k \\
[[y_2]] &\to \sum_{i=0}^{5p-1} \mathrm{link}(\widehat{\epsilon^i L}, (\widehat{y_1 + L^\perp}))\epsilon^i l + \sum_{i=0}^{4p-1} \mathrm{link}(\epsilon^i(\widehat{L \cap \epsilon^p L}), (\widehat{y_1 + L^\perp}))\epsilon^i k = l.
\end{aligned}$$

Since in this case $[c'_{\mathbb{Z}_n}(h)(d_3)] = 2[[y_1]] - 2[[y_2]] + K$ we have that $[c'_{\mathbb{Z}_n}(h)(d_3)] \in H_2(M;\mathbb{Z})_{\mathbb{Z}_n} \cong \mathbb{Z} \oplus \mathbb{Z}_5$ is $2 \cdot$ (generator of the second summand) and so must be different from zero. Therefore, there are no $\mathbb{Z}_n$-maps $F : S^3 \to M(\alpha)$. ■

Theorem 19 *A partial answer to the problem 3 for the whole group Q_{4n} is the following.*

(A) If $\alpha = (p,p,p,p)$ or $\alpha = (p,p,p,2p)$, then there are no Q_{4n}-maps $F : S^3 \to M(\alpha)$.

(B) If α is not of the form $(p,q,p+q,p+q)$, $p \neq q$, or $(p,q,r,p+q)$, $r \notin \{p,p+q\}$, or $(p,p,q,q+r)$, then there exists a Q_{4n}-map $F : S^3 \to M(\alpha)$.

(C) If $\alpha = (p,q,p+q,p+q)$, $p \neq q$, or $\alpha = (p,q,r,p+q)$, $r \notin \{p,p+q\}$, or $\alpha = (p,p,q,q+r)$, the methods of the paper are inconclusive and a further analysis is needed.

Proof. The proof goes along the lines of the proof of the preceding theorem. The case (A) is the direct consequence of the preceding theorem, but it can be also derived by a direct calculation interpreting the Q_{4n} obstruction cocycle in the appropriate group of coinvariants. The case (B) follows from the following facts:

(1) the cohomology class of the obstruction cocycle $[c'_{Q_{4n}}(h)(d_3)]$ is divisible by 4 (Theorem 11),

(2) the cohomology class of the obstruction cocycle $[c'_{Q_{4n}}(h)(d_3)]$ is a torsion element (Proposition 7), and

(3) the only torsion group which appears as a summand in the coinvariant group is $\mathbb{Z}_2$ (Theorem 17).
■

5 Appendix

5.1 The geometry of the group Q_{4n}

It is of utmost importance for computations to describe an economical and geometrically transparent G-invariant, CW-structure on a given G-manifold. In the case of the generalized quaternion group Q_{4n} acting on the sphere S^3 of all unit quaternions, such a structure is described by the following construction.

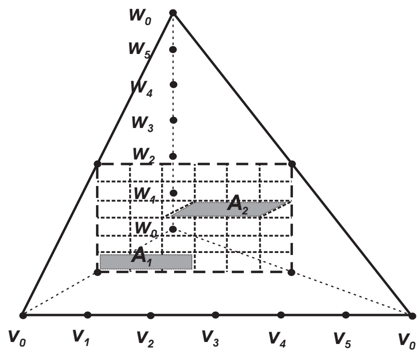

Figure 6: The sphere $S^3 = P_{2n}^{(1)} * P_{2n}^{(2)}$ in the case $n = 3$

We have already met in Section 3.1 the join decomposition $S^3 = P_{2n}^{(1)} * P_{2n}^{(2)}$ of the 3-sphere. The case $n = 3$ is depicted on the Figure 6 where $S^3 = S^1 * S^1 = [0,1] * [0,1]/\approx$ and "$\approx$" is the equivalence relation on the 3-simplex $\sigma^3 := [0,1] * [0,1]$ arising from the identification $S^1 \cong [0,1]/0 \sim 1$. The (big) rectangle represents the torus $T^2 := \{\frac{1}{2}x + \frac{1}{2}y \mid x, y \in S^1\}$ subdivided into small rectangles which are in 1–1 correspondence with simplices $[v_i, v_{i+1}; w_j, w_{j+1}]$. Since each point $a = \frac{1}{2}x + \frac{1}{2}y$ of this rectangle uniquely determines the line segment $[x, y] \subset S^3$, each subset $A \subset T^2$ determines uniquely a set $\hat{A} \subset S^3$ such that $\hat{A} \cap T^2 = A$. Let $A_1 := \cup_{j=0}^{n-1}[v_i, v_{i+1}] \times [w_0, w_1]$ be a rectangle in T^2 and A_2 a closely related "skew" rectangle, see the Figure 6. Then both $\hat{A}_1$ and $\hat{A}_2$ are fundamental domains for the Q_{4n}-action on S^3, both inducing Q_{4n}-invariant CW-structures on S^3. The second of these two structures is simpler having only one Q_{4n}-cell a in dimension 0, two Q_{4n}-cells b and b' in dimension 1, two Q_{4n}-cells c and c' in dimension 2, and one Q_{4n}-cell e in dimension 3. The reader can easily check that the associated cellular chain complex $\{D_i(S^3, \mathbb{Z})\}_{i=0}^3$ of Q_{4n}-modules coincides with the well known minimal resolution of $\mathbb{Z}$ by free Q_{4n}-modules described on p. 253 of the classical monograph [8]. More precisely

$$0 \to Z(Q_{4n})e \xrightarrow{\partial} Z(Q_{4n})c \oplus Z(Q_{4n})c' \xrightarrow{\partial} Z(Q_{4n})b \oplus Z(Q_{4n})b' \xrightarrow{\partial} Z(Q_{4n})e \to 0$$

where

$$\begin{array}{lll} \partial e = (\epsilon - 1)c - (\epsilon j - 1)c'; & \partial c = Nb - (j+1)b'; & \partial c' = (\epsilon j + 1)b + (\epsilon - 1)b' \\ & \partial b = (\epsilon - 1)a & \partial b' = (j-1)a \end{array}$$

and $N = 1 + \epsilon + \ldots + \epsilon^{n-1} \in \mathbb{Z}(Q_{4n})$. On applying the functor $\mathrm{Hom}_{Q_{4n}}(\cdot, N))$, where $N := H_2(M; \mathbb{Z})$ is a Q_{4n}-module, one obtains the sequence,

$$0 \longleftarrow H_2(M;\mathbb{Z}) \xleftarrow{\Gamma} H_2(M;\mathbb{Z}) \oplus H_2(M;\mathbb{Z}) \leftarrow H_2(M;\mathbb{Z}) \oplus H_2(M;\mathbb{Z}) \longleftarrow H_2(M;\mathbb{Z}) \longleftarrow 0$$

where $\Gamma(p, q) = (\epsilon - 1)p - (\epsilon j - 1)q$ for $p, q \in H_2(M; \mathbb{Z})$. Now it is not hard to prove

$$H^3_{Q_{4n}}(S^3, H_2(M;\mathbb{Z})) = H_2(M;\mathbb{Z})/\mathrm{Im}\Gamma \cong H_2(M;\mathbb{Z})_{Q_{4n}}$$

where $H_2(M; \mathbb{Z})_{Q_{4n}}$ is a group of coinvariants of the Q_{4n}-module $H_2(M; \mathbb{Z})$, [7]. Alternatively, the last result can be seen as a consequence of the equivariant Poincaré duality, [24].

The complex associated with the fundamental domain A_1, Figure 6, is not as economical as the complex based on the fundamental domain A_2. Note however that for our purposes this is not a serious disadvantage. Indeed, all we need to know is the intersection of $e_1 = \mathrm{int}(A_1)$ with the singular set $h^{-1}(\cup\mathcal{A}(\alpha))$, for a generic Q_{4n}-equivariant map h, and interpret the answer as an element of the group $H_2(M; \mathbb{Z})_{Q_{4n}}$, cf. Sections 4.1.1 and 4.1.2. On the other hand, the choice of A_1 has an advantage that its top cell $e_1 = A_1$ is a union of simplices from the original simplicial structure on S^3, see Figure 6, which considerably simplifies the computations performed in Section 4.1.2.

5.2 Change of the group

Let X be a left G-space and $H \lhd G$ be a normal subgroup. Then X/H is a G/H-space with the action given by $gH(Hx) = H(gx)$. A G/H-space Z can be also seen as a G-space via the quotient homomorphism $\pi : G \to G/H$ where for $g \in G$ and $z \in Z$, $g \cdot z := \pi(g)z$.

Proposition 20 *Suppose that X and Z are G-spaces and $H \lhd G$ is a normal subgroup of G which acts trivially on Z. Then there exists a G-map $\alpha : X \to Z$ if and only if there exists a G/H-map $\beta : X/H \to Z$, where X/G and $Z/H = Z$ are interpreted as G/H-spaces.*

Proof. $\Leftarrow$: Observe that the quotient map $p : X \to X/H$ is a G-map and that the G/H-space X/H is a G-space via the homomorphism $\pi : G \to G/H$. If $\beta : X/H \to Z$ is a G/H-map, it is also a G-map and the composition $\alpha := p \circ \beta : X \to Z$ is a G-map.

$\Rightarrow$: Let $\alpha : X \to Z$ be a G-map. Then, since H-acts trivially on Z, there is a factorization $\alpha = \beta \circ p$ for some $\beta : X/H \to Z$. We check that β is a G/H-map,

$$\beta(gH \cdot Hx) = \alpha(gx) = g\alpha(x) = g\beta(Hx) = (gH)\beta(Hx).$$

■

5.3 Modifying free G-action on S^3

(A) Suppose that $\gamma_i : G \times S^3 \to S^3, \gamma_i(g,x) = g \cdot_i x$, $i = 1,2$ are two actions of a finite group G on the 3-sphere S^3, and assume that the action γ_1 is free. Then there exists a G-equivariant map $f : S^3 \to S^3$ between these two actions, i.e.

$$(\forall g \in G)\,(\forall x \in S^3)\ f(g \cdot_1 x) = g \cdot_2 f(x).$$

(B) If in addition the action γ_2 is free, then for any G-space Z there exists a γ_1-map $f : S^3 \to Z$ if and only if such a map exists for the γ_2 action.

Proof. (A) The proof is routine and relies on the fact that S^3 is a 2-connected, γ_1-free, CW-complex so there are no obstructions to extend equivariantly a map defined on the 0-skeleton of S^3. The statement (B) is a direct consequence of (A). ■

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